Algebras associated with Pseudo Reflection Groups: A Generalization of Brauer Algebras

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Abstract

We present a way to associate an algebra $B_G(\Upsilon)$ with every pseudo reflection group G. When G is a Coxeter group of simply-laced type we show $B_G(\Upsilon)$ is isomorphic to the generalized Brauer algebra of simply-laced type introduced by Cohen,Gijsbers and Wales[10]. We prove $B_G(\Upsilon)$ has a cellular structure and be semisimple for generic parameters when G is a rank 2 Coxeter group. In the process of construction we introduce a Cherednik type connection for BMW algebras and a generalization of Lawrence-Krammer representation to complex braid groups associated with all pseudo reflection groups.

1 Introduction

The Brauer algebras $B_n(\tau)$ were introduced by Brauer in 1930's, motivated by the purpose of creating a Schur-Weyl duality for orthogonal groups and symplectic groups. Roughly speaking the Brauer algebra $B_n(\tau)$ is an algebra containing the group algebra of symmetric group kS_n, and sharing many important properties with kS_n. First, $B_n(\tau)$ is semisimple for generic τ as proved in [27], whence their irreducible representations are also labeled by Young diagrams. Secondly just as kS_n has Hecke algebra as a deformation , which provide a way to define the HOMFLY polynomial invariants for Links, Brauer algebras have a natural deformation named Birman-Murakami-Wenzl algebras(also called BMW algebras),which dominate the Kauffman polynomial invariants in similar way [3][23]. The Brauer algebras also admit cellular structures like kS_n in the sense of Graham and Lehrer[15].

As many objects related to Lie theory, In [10] Cohen, Gijsbers and Wales defined BMW algebras and Brauer algebras for simply-laced Coxeter groups. Those new algebras have almost all important algebraic properties of $B_n(\tau)$. They were shown to be semisimple for D_n type for generic parameters, and admitting cellular structures ([11]). In [10] the author ask whether there exist a definition of generalized BMW algebras for non-simply laced Coxeter groups. In another way in [16] Haring-Oldenburg defined Cyclotomic BMW algebras and their degenerate version Cyclotomic Brauer algebras. These algebras were studied in many subsequent papers.

In this paper we introduce an algebra $B_G(\Upsilon)$ for every Coxeter group and every pseudo reflection group G, where Υ is a set of parameters. These algebras have the following properties .

If G is finite then $B_G(\Upsilon)$ is a finite dimensional algebra containing the group ring kG naturally.

There exist a flat formal connection which can deform every representation $B_G(\Upsilon)$ to a one-parameter family of representations of A_G, the braid group associated with G. One of these representations is the generalized Lawrence-Krammer representation.

When G is a Coxeter group of simply-laced type, $B_G(\Upsilon)$ is isomorphic to the generalized Brauer algebra introduced in [10]. When G is a pseudo reflection group of $G(m,1,n)$ type, the cyclotomic Brauer algebra introduced in [16] is isomorphic to a natural quotient of our algebra $B_G(\Upsilon)$.

When G is a rank 2 Coxeter group the algebra $B_G(\Upsilon)$ is semisimple for generic Υ and admit a cellular structure(Thm 5.2,Prop5.2).

Two major ingredients in the construction of $B_G(\Upsilon)$ are Cherednik type connections for BMW algebras and the generalized Lawrence-Krammer representations of Cohen,Wales [9] and Marin [22].

Let $H_n(q)$ be the Hecke algebra of the symmetric group S_n. Let

$$Y_n = \{(z_1,\cdots,z_n) \in \mathbb{C}^n | z_i \neq z_j, \text{ for any } i \neq j\}.$$

It is well-known that for generic q the set of irreducible representations of $H_n(q)$ is in one to one correspondence with the set of irreducible representations of S_n. This correspondence has a geometric description. The Cherednik connection describe how a representation of S_n can be deformed to a representation of $H_n(q)$.

Let $\omega_{i,j} = d\log(z_i - z_j)$ for $i < j$ be closed forms on Y_n. The Cherednik connection is a formal flat connection

$$\Omega_n = \kappa \sum_{i<j} s_{i,j}\omega_{i,j}$$

defined on the bundle $Y_n \times \mathbb{C}S_n$. Where $s_{i,j} \in S_n$ is the (i,j) permutation.

Cyclotomic Hecke algebras also have such flat connections as shown in [5]. Let G be a pseudo reflection group acting on a complex linear space E, let $H_G(\bar{q})$ be the associated cyclotomic Hecke algebra. We denote the set of reflection hyperplanes of G as $\{H_\nu\}_{\nu\in P}$, and denote the set of pseudo reflections of G as R. Define $\nu : R \to P$ by requesting the reflection hyperplane of s to be $H_{\nu(s)}$. Denote $M_G = E \setminus \cup_{i\in P} H_i$ being the complementary space of reflection hyperplanes. By Steinberg theorem [26] the induced action of G on M_G is free. Complex braid group A_G associated with G is defined as $\pi_1(M_G/G)$. For any $\nu \in P$ let f_ν be any nonzero linear function on E with kernel H_ν, let $\omega_\nu = d\log f_\nu$. The Cherednik type connection of G is defined on the bundle $M_G \times \mathbb{C}G$ as follows

$$\Omega_G = \sum_{\nu\in P} \kappa(\sum_{s\in R, \nu(s)=\nu} \mu_s s)\omega_\nu.$$

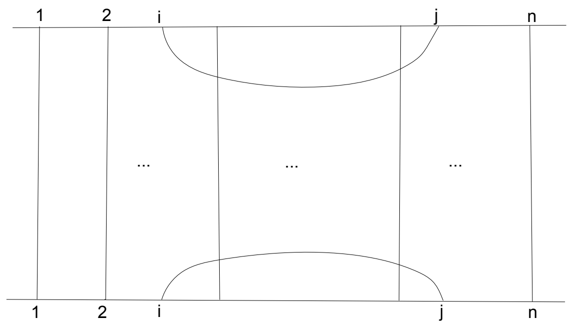
Where $\{\mu_s\}_{s\in R}$ is a set of parameters satisfying $\mu_{s_1} = \mu_{s_2}$ if s_1 is conjugate to s_2. Suppose (V,ρ) is linear representation of G, then we have a flat connection on bundle $M_G \times V$:

$$\rho(\Omega_G) = \sum_{\nu\in P} \kappa(\sum_{s\in R, \nu(s)=\nu} \mu_s \rho(s))\omega_\nu.$$

$\rho(\Omega_G)$ induce a flat connection on the quotient bundle $M_G \times_G V$ which define a one parameter class of representations of A_G through monodromy. These monodromy representations factor through $H_G(\bar{q})$ for suitable $\bar{q}$.[5]

It is known that for generic parameters the set of irreducible representations of BMW algebras is also in one to one correspondence with the set of irreducible representations of the Brauer algebra $B_n(\tau)$. It is reasonable to expect that there exist Cherednik type connection for BMW algebras.

We find the formal connection $\bar{\Omega}_n = \kappa \sum_{i<j}(s_{i,j} - e_{i,j})\omega_{i,j}$ defined on $Y_n \times B_n(\tau)$ is just what we want. It is flat and can deform every representation of Brauer algebra to a representation of BMW algebra(Thm 3.2). Where $e_{i,j} \in B_n(\tau)$ is the element described by the following graph.



$e_{i,j}$

Now we see the deformable property of $\mathbb{C}G$ and $B_n(\tau)$ embody in the existence of a flat formal connection describing the deformation. We hope generalized Brauer algebras should be deformable and have such flat connections as well. The concise form of above flat connection motivate us to imagine that $B_G(\Upsilon)$ should be generated by elements in G along with a set of special elements $\{e_i\}_{i\in P}$, such that the following formal connection $\bar{\Omega}_G$ on $M_G \times B_G(\Upsilon)$ is flat and G-equivariant.

$$\bar{\Omega}_G = \sum_{\nu\in P} \kappa(\sum_{s\in R, \nu(s)=\nu} \mu_s s - e_\nu)\omega_\nu.$$

If all pseudo reflections in G have rank 2, and if we set all parameters $\mu_s = 1$, then the connection become: $\bar{\Omega}_G = \sum_{\nu\in P} \kappa(s_\nu - e_\nu)\omega_\nu$, which has a form similar with $\bar{\Omega}_n$.

The flatness condition give some relations between s_i's and e_i's. But these relations are too coarse to produce an interesting finite dimensional algebra.

Annother common feature of BMW algebras of simply-laced types is that they all contain the generalized Lawrence-Krammer representations(from now on we call them by LK representations) defined in [9](see[10]). Infinitsiml version of LK representations are found by Marin[21]. They are described by some concise flat connections. Marin then generalized these flat connections to all complex reflection groups(pseudo reflection groups whose pseudo reflections all have rank 2), thus generalize LK representations to all complex reflection groups [22](see 5.2). In section 4 We show LK representations can actually be generalized to all pseudo reflection groups and have a very simple characterization.

Let $V_G = \mathbb{C}\{v_i\}_{i\in P}$ be a vector space with a basis in 1 to 1 correspondence with $\{H_i\}_{i\in P}$, the set of reflection hyperplanes of G. Since G permutes the set of reflection hyperplanes, there is a natural representation $\iota : G \to GL(V_G)$.

Theorem 4.2 *For every* $i \in P$, *suppose* $p_i \in \mathrm{End}(V_G)$ *is a projection to* $\mathbb{C}v_i$. *Explicitly suppose* $p_i(v_j) = \alpha_{i,j}v_i$ *for* $j \neq i$; $p_i(v_i) = m_i v_i$. *Then the following connection on* $M_G \times V_G$, $\Omega_{LK} = \sum_{i\in P} \kappa(\sum_{s:\nu(s)=i} \mu_s \iota(s) - p_i)\omega_i$ *is flat and* G*-equivariant if and only if*

$\alpha_{i,j} = \sum_{s:\iota(s)(v_j)=v_i} \mu_s$, *and* $m_i = m_j$ *if there exist* $w \in G$ *such that* $w(H_i) = H_j$.

When Ω_{LK} satisfy the G invariant and flatness condition in Theorem 4.2, it induce a flat connection $\bar{\Omega}_{LK}$ on the quotient bundle $M_G \times_G V_G$. We define the generalized LK representation of A_G as the monodromy representation of $\bar{\Omega}_{LK}$. When $\mu_s = 1$ for all s, the connection Ω_{LK} become the flat connection of Marin [22].

To find more relations for $B_G(\Upsilon)$ now we make annother request that all generalized BMW algebras should contain generalized LK representations. This condition can be presented as: there exist some representation (V, ρ) of $B_G(\Upsilon)$, such that the connection $\rho(\bar{\Omega}_G)$ coincide with Ω_{LK}. This request give us both restriction and indication on relations we are looking for. We find the algebra defined in Definition 5.1 satisfies these requests perfectly.

Definition 5.1 *The algebra* $B_G(\Upsilon)$ *associated with pseudo reflection group* (V, G) *is generated by* $\{\bar{w}\}_{w\in G} \cup \{e_i\}_{i\in P}$, *submitting to the following relations. Here* P *is the index set of reflection hyperplanes of* G *as defined before.*

(0) $\bar{w}_1\bar{w}_2 = \bar{w}_3$ *if* $w_1w_2 = w_3$;

(1) $\bar{s}_ie_i = e_i\bar{s}_i = e_i$, *for* $i \in P$;

(2) $e_i^2 = m_ie_i$;

(3) $\bar{w}e_j = e_i\bar{w}$, *if* $w \in G$ *satisfies* $w(H_j) = H_i$;

(4) $e_ie_j = e_je_i$, *if* $\{ k \in P| \ H_k \supset H_i \cap H_j\} = \{i, j\}$;

(5) $e_ie_j = (\sum_{s\in R(i,j)} \mu_s s)e_j$, *if* $\{k \in P|H_k \supset H_i \cap H_j\} \neq \{i, j\}$, *and* $R(i, j) \neq \emptyset$.

(6) $e_ie_j = 0$, *if* $\{k \in I|H_k \supset H_i \cap H_j\} \neq \{i, j\}$ *and* $R(i, j) = \emptyset$.

where $R(i, j) = \{s \in R|s(H_j) = H_i\}$. s_i *is any pseudo reflection fixing* H_i. *Constants* $\{\mu_s\}_{s\in R}$ *satisfy* $\mu_{s_1} = \mu_{s_2}$ *if* s_1 *is conjugate to* s_2 . *Constants* $\{m_i\}_{1\in P}$ *satisfy* $m_i = m_j$ *if there exists* $w \in G$ *such that* $w(H_i) = H_j$. *The symbol* Υ *means data* $\{\mu_s\} \cup \{m_i\}$.

For these algebras $B_G(\Upsilon)$, we prove they are finite dimensional(Thm 5.1), the connection $\bar{\Omega}_G$ is G-invariant and flat (Prop 5.1). And It is readily to see it could realize the connection Ω_{LK}. We hope these algebras satisfy other important properties of Brauer algebra: Semisimple for generic parameters; have cellular structures; can be deformed to certain generalized BMW algebras. Existence of the connection Ω_{Br} seems supporting the last property. Section 5.2 is devoted to study of cases when G are dihedral groups. We prove they are semisimple for generic parameters(Thm 5.2,Thm 5.5), and have cellular structures(Thm 5.3, Prop 5.2). We hope these two properties hold for all $B_G(\gamma)$.

In section 5.3 we construct canonical presentations of $B_G(\Upsilon)$ when G is a Coxeter group of finite type (Thm 5.10), when G is of simply-laced type, it can be seen from this presentation that $B_G(\Upsilon)$ coincide with the generalized Brauer algebra in [9]. When G is a pseudo reflection group of type $G(m, 1, n)$ we also find a canonical presentation for $B_G(\Upsilon)$ (Thm 5.11), from which we see the cyclotomic Brauer algebra introduced in [12] is isomorphic to a quotient of $B_G(\Upsilon)$ by letting some element e_i to be zero.

In the last section 5.4 we discuss the uniqueness of $B_G(\Upsilon)$. If we replace (6) in definition 5.1 with a milder relation

$$(6)^{'}: e_i e_j = e_j e_i, \mathrm{if} \{k \in I | H_k \supset H_i \cap H_j\} \neq \{i,j\} \mathrm{and} R(i,j) = \emptyset$$

and keep other relations, the resulted algebra $\hat{B}_G(\Upsilon)$ satisfies all above mentioned properties of $B_G(\Upsilon)$ except semisimplity and existence of a cellular structure. Proposition 5.4 shows when G is a rank 2 Coxeter groups, for generic Υ the relation $(6)^{'}$ degenerate to relation (6). As we know BMW algebra have the same dimension for all parameters, proposition 5.4 seems proving $\hat{B}_G(\Upsilon)$ isn't a good choice.

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2 Preliminaries

2.1 Brauer algebras and BMW algebras

Brauer algebra $B_n(\tau)$ is a graphic algebra in the sense that it has a basis consisting of elements presented by graphs, and the relations between them can be described through graphs. $B_n(\tau)$ has a canonical presentation with generators $s_1, \cdots, s_{n-1}$, $e_1, \cdots, e_{n-1}$ and relations listed in table 1. $\mathcal{B}_n(\tau)$ has a natural deformation discovered by Birman, Murakami, Wenzl which are now called BMW algebras [3][23]. These algebras support a Markov trace which gives the Kauffman polynomial invariants of Links. We denote these BMW algebras as $B_n(\tau, l)$. Where l is a parameter of deformation. There is $B_n(\tau) \cong B_n(\tau, 1)$. We list generators and relations of $B_n(\tau)$ and $B_n(\tau, l)$ in table 1 according to [9]. Where $m = \frac{l - l^{-1}}{1-\tau}$.

	$B_n(\tau)$	$B_n(\tau, l)$
Generators	$s_1, \cdots, s_{n-1}; e_1, \cdots, e_{n-1}$	$X_1, \cdots, X_{n-1}; E_1, \cdots, E_{n-1}$
Relations	$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$	$X_i X_{i+1} X_i = X_{i+1} X_i X_{i+1}$
	for $1 \leq i \leq n-2$;	for $1 \leq i \leq n-2$;
	$s_i s_{i-1} e_i = e_{i-1} s_i s_{i-1}$	$X_i X_{i-1} E_i = E_{i-1} X_i X_{i-1}$
	for $2 \leq i \leq n-1$;	for $2 \leq i \leq n-1$;
	$s_i s_{i+1} e_i = e_{i+1} s_i s_{i+1}$	$X_i X_{i+1} E_i = E_{i+1} X_i X_{i+1}$
	for $1 \leq i \leq n-2$;	for $1 \leq i \leq n-2$;
	$s_i s_j = s_j s_i$, $\lvert i-j \rvert \geq 2$;	$X_i X_j = X_j X_i, \lvert i-j \rvert \geq 2$;
	$s_i^2 = 1$, for all i;	$l(X_i^2 + mX_i - 1) = mE_i$, for all i;
	$s_i e_i = e_i$, for all i;	$X_i E_i = l^{-1} E_i$, for all i;
	$e_i s_{i+1} e_i = e_i, 1 \leq i \leq n-2$;	$E_i X_{i+1} E_i = l E_i, 1 \leq i \leq n-2$;
	$e_i s_{i-1} e_i = e_i, 2 \leq i \leq n-1$;	$E_i X_{i-1} E_i = l E_i, 2 \leq i \leq n-1$;
	$s_i e_j = e_j s_i, \lvert i-j \rvert > 1$;	$X_i E_j = E_j X_i, \lvert i-j \rvert > 1$;
	$e_i^2 = \tau e_i$, for all i.	$E_i^2 = \tau E_i$.

The structure of Brauer algebras and BMW algebras are studied in [3][27]. They have the following basic properties.

Theorem (Wenzl) Let the ground ring be a field of character 0, then $\mathcal{B}_n(\tau)$ is semisimple if and only if $\tau \notin \mathbb{Z}$ or $\tau \in \mathbb{Z}$ and $\tau > n$.

Many algebras related to Lie theory including Hecke algebras, BMW algebras are cellular algebras defined as follows [15]. In the same paper Graham and Lehrer construct cellular structure for BMW algebras and Brauer algebrsa.

Definition (Graham, Lehrer)[15] A cellular algebra over R is an associative algebra A, together with cell datum $(\Lambda, M, C, *)$ where

- (C1) Λ is a partially ordered set and for each $\lambda \in \Lambda$,$M(\lambda)$ is a finite set such that$C : \cap_{\lambda\in\Lambda} M(\lambda) \times M(\lambda) \longrightarrow A$ is an injective map with image an R-basis of A.

- (C2) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, write $C(S,T) = C^{\lambda}_{S,T} \in A$. Then $*$ is an R-linear anti-involution of A such that $*(C^{\lambda}_{S,T}) = C^{\lambda}_{T,S}$.

- (C3) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ then for any element $a \in A$ we have
  $aC^{\lambda}_{S,T} \equiv \sum_{S'\in M(\lambda)} r_a(S', S) C^{\lambda}_{S',T} (\mathrm{mod} A(< \lambda))$
  Where $r_a(S', S) \in R$ is independent of T and where $A(< \lambda)$ is the R-submodule of A generated by $\{C^{\mu}_{S'',T''} | \mu < \lambda; S'', T'' \in M(\mu)\}$..

In table 1 we observe the presentation can be related to type A_{n-1} Dynkin diagram in the following way. Every node p_i ($1 \le i \le n-1$) of the Dynkin diagram corresponds to a pair of generators s_i, e_i. Between s_i, e_i there are relations $s_i^2 = 1$, $e_i^2 = \tau e_i$ and $s_i e_i = e_i s_i = e_i$. There are two pattern of relations between s_i, e_i and s_j, e_j. When p_i is connected to p_j by an edge, equivalently $i = j \pm 1$, there are relations $s_i s_j s_i = s_j s_i s_j$, $e_i s_j e_i = e_i$, $e_j s_i e_j = e_j$ and $s_i s_j e_i = e_j s_i s_j$. When p_i isn't connected to p_j, there are relations $s_i s_j = s_j s_i$, $e_i e_j = e_j e_i$, $s_i e_j = e_j s_i$. These are all relations for $B_n(\tau)$.

A simply-laced Dynkin diagram is a Dynkin diagram whose edges are simply-laced. Finite type simply-laced Dynkin diagram consists ADE type Dynkin diangrams. For every such Dynkin diagram Γ, above discussion shows how to define algebras similar with $B_n(\tau)$ and $B_n(\tau, l)$. This is due to Cohen, Gijsbers and Wales. In [10] they define algebra $B_\Gamma(\tau)$ and algebra $B_\Gamma(\tau, l)$ with generators and relations as in table 2. These are generalization of Brauer algebra and BMW algebra to simply-laced type root systems (or Coxeter groups).Let I be the set of nodes of Γ. When $i, j \in I$ are connected by an edge we write $i \sim j$. Otherwise we write $i \nsim j$. Set $m = \frac{l - l^{-1}}{1-\tau}$.

	$B_\Gamma(\tau)$	$B_\Gamma(\tau, l)$
Generators	$s_i (i \in I)$; $e_i (i \in I)$	$X_i (i \in I)$; $E_i (i \in I)$
Relations	$s_i s_j s_i = s_j s_i s_j$, if $i \sim j$;	$X_i X_j X_i = X_j X_i X_j$, if $i \sim j$;
	$s_i s_j e_i = e_j s_i s_j$ if $i \sim j$;	$X_i X_j E_i = E_j X_i X_j$ if $i \sim j$;
	$s_i s_j = s_j s_i$,if $i \nsim j$;	$X_i X_j = X_j X_i$, if $i \nsim j$;
	$s_i^2 = 1$, for all i;	$l(X_i^2 + mX_i - 1) = mE_i$, for all i;
	$s_i e_i = e_i$, for all i;	$X_i E_i = l^{-1} E_i$, for all i;
	$e_i s_j e_i = e_i$, if $i \sim j$;	$E_i X_j E_i = l E_i$, ifi $\sim j$;
	$s_i e_j = e_j s_i$, if $i \nsim j$;	$X_i E_j = E_j X_i$, ifi $\nsim j$;
	$e_i^2 = \tau e_i$, for all i.	$E_i^2 = \tau E_i$.

These generalized Brauer algebras have no graph to representing their elements any more, but they have almost all important algebraic properties of Brauer algebras.

2.2 Pseudo reflection groups, Complex braid groups and Hecke algebras

Let V be a complex linear space. An element s in $GL(V)$ is called a pseudo reflection if it can be presented as $diag(\xi, 1, \cdots, 1)$ where ξ is a root of unit. If ξ is -1 then s is simply called a reflection. $G \subset GL(V)$ is called a pseudo reflection group if it is generated by pseudo reflections. If G is generated by reflections then it is called a complex reflection group. When V is an irreducible representation of G, (V, G) is called an irreducible pseudo reflection group. Every pseudo reflection group is isomorphic to direct product of some irreducible factors. Isomorphism class of irreducible pseudo reflection groups are classified by Shephard-Todd [25]. They consists of a infinite family $\{ G(m, p, n) \}$ ($n \leq 1$,$m \leq 2$,$p|m$) and 34 exceptional ones. Like reflection groups, one can associate a braid group and a Hecke algebra with every pseudo reflection group as follows.

Let (V, G) be a pseudo reflection group. Let $R = \{s_i\}_{i \in I}$ be the set of pseudo reflections contained in G. For a pseudo reflection s, the reflection hyperplane H_s is defined as the 1 characteristic space of s, which is a codimension 1 subspace of V. Let $M_G = V - \cup_{s \in R} H_s$ be the complementary space of all reflection hyperplanes in V. A theorem of Steinberg says the action of G on M_G is free. The braid group $A_G$associated with G is defined as $\pi_1(M_G/G)$. It's subgroup $P_G = \pi_1(M_G)$ is called the pure braid group associated with G. There is a short exact sequence $1 \to P_G \to A_G \to G \to 1$.

As in [2][5], a Hecke algebra $H_G(\bar{\lambda})$can be associated with every pseudo reflection group G, where $\bar{\lambda}$ is a set of parameters. The Hecke algebra $H_G(\bar{\lambda})$ is a quotient algebra of the group algebra $\mathbb{C}B_G$. It's dimensional equals $|G|$. For generic $\bar{\lambda}$, $H_G(\bar{\lambda})$ is a semisimple algebra whose irreducible representations are in one to one correspondence with those of G in a natural way. This correspondence can be described by the following Cherednik type connection.

Suppose the set of reflection hyperplanes of G is $\{H_\nu\}_{\nu \in P}$ and the set of pseudo reflections in G is R. Define a map $\nu : R \to P$ by requesting the reflection hyperplane of s to be $H_{\nu(s)}$. There is a natural action of G on P induced by the action of G on the set of reflection hyperplanes. Now for every reflection hyperplane H_ν, chose a linear form f_ν with kernel H_ν. Define a closed 1 form on M_G as $\omega_\nu = d \log f_\nu$. Suppose $\{\mu_s\}_{s \in R}$ is a set of constants satisfying the condition: $\mu_{s_1} = \mu_{s_2}$ if s_1 is conjugate to s_2.

Proposition 2.1. [5] *The formal connection* $\Omega_G = \kappa \sum_{\nu \in P} (\sum_{s \in R, \nu(s) = \nu} \mu_s s) \omega_\nu$ *on* $M_G \times \mathbb{C}G$ *is flat and* G*- invariant.*

Here for convenience we use a slightly different form of Ω_G. Suppose (E, ρ) is a linear representation of G . By above proposition $\rho(\Omega_G) = \kappa \sum_{\nu \in P} (\sum_{s \in R, \nu(s) = \nu}) \mu_s \rho(s) \omega_\nu$ defines a flat connection on the bundle $M_G \times E$. It can induce a flat connection $\bar{\Omega}_\rho$ on the quotient bundle $M_G \times_G E$ because of G invariance of $\rho(\Omega_G)$. By taking monodromy a family of representations of B_G parameterized by κ, μ_s's are obtained. It is proved in [5] that these monodromy representations factor through $H_G(\bar{\lambda})$ for suitable choice of $\bar{\lambda}$.

We give a simple setting for G-invariance of connections. Let A be an algebra with unit 1, $A^{\times}$ be the set of invertible elements in A, which form a group under the multiplication. Let $\varphi : G \to A^{\times}$ be a morphism of groups. Let $\rho : A \to End(E)$ be a representation of A such that $\rho(1) = id_E$. Then $\varphi \circ \rho : G \to End(E)$ is a representation of G, we denote it as $\bar{\rho}$.

The group G acts on the bundle $E \times M_G$ as: $g \cdot (v, p) = (\bar{\rho}(g)(v), g \cdot p)$, for $g \in G, v \in E$, and $p \in M_G$. The quotient $E \times M_G/G$ is a linear bundle on M_G/G. And G acts on $A \times M_G$ as: $g \cdot (x, p) = (\varphi(g)x\varphi(g)^{-1}, g \cdot p)$, for $g \in G$, $x \in A$, and $p \in M_G$. Then the quotient $A \times M_G/G$ is a bundle of algebras on M_G.

Now Let $\Omega = \sum_{i \in P} X_i \omega_i$ be a connection on $A \times M_G$, where $X_i \in A$. Then $\Omega_\rho = \sum_{i \in P} \rho(X_i)\omega_i$ is a connection on the bundle $E \times M_G$. We have

Proposition 2.2 (G-invariance). *The connection Ω induce a connection on the quotient bundle if and only if $X_{w(i)} = \varphi(w)X_i\varphi(w)^{-1}$, for any $i \in P$, $w \in G$.*

The connection Ω_ρ induce a connection on the quotient bundle $E \times M_G$ if and only if $\rho(X_{w(i)}) = \bar{\rho}(w)\rho(X_i)\bar{\rho}(w)^{-1}$, for any $i \in P$, $w \in G$.

When Ω and Ω_ρ are flat connections, the equation of flat sections for Ω is: $dF = \Omega F$, Where $F : M_G \to A$ is a section. The equation of flat sections for Ω_ρ is $d\Phi = \Omega_\rho \Phi$, where $\Phi : M_G \to E$ is a section. Flat sections of these two bundles are connected as follows. Let $U \subset M_G$ be some region, and $F : U \to A$ be a local flat section. Then for any $v \in E$, $F \cdot v : U \to E$ is a local flat section on $E \times M_G$.

We consider the monodromy operators when Ω and Ω_ρ are flat. Let $\pi : M_G \to M_G/G$ be the quotient map. Chose a base point $p_0 \in M_G$, and chose $\bar{p_0} = \pi(p_0)$ as the base point of M_G/G. We define some special path classes named generators of monodromy as follows. This presentation is borrowed from [5].

Let H_v be a reflection hyperplane, $q \in H_v \setminus \sum_{u \neq v} H_u$ be a generic point.Let $L_v = im(s_v - Id_V)$, which is a complex line and we have $V = H_v \oplus L_v$. For $p \in V$, we set $p = pr_v(p) + pr_v^{\perp}(p)$ with $pr_v(p) \in H_v$ and $pr_v^{\perp}(p) \in L_v$. Thus, we have

$$s_v(p) = e^{\frac{2\pi\sqrt{-1}}{m_v}} pr_v^{\perp}(p) + pr_v(p).$$

let $\gamma : [0, 1] \to V$ be a path in V starting from p_0 and ending at q,such that $\gamma(t) \in M_G$ for $t \neq 1$. For $\epsilon \in (0, 1]$, we denote the partial path $\gamma|_{[0,1-\epsilon]} : [0, 1 - \epsilon] \to V$ as γ_ϵ, and the point $\gamma(1 - \epsilon)$ as p_ϵ.

For $p \in V$, we define a path $\sigma_{v,p}$ from p to $s_v(p)$ as follows:

$$\sigma_{v,p} : [0, 1] \to V, \ t \mapsto e^{\frac{2\pi\sqrt{-1}t}{m_v}} pr_v^{\perp}(p) + pr_v(p).$$

Denote the composed path $s_v(\gamma_\epsilon^{-1}) \circ \sigma_{v,p_\epsilon} \circ \gamma_\epsilon$ as $\sigma_{v,\gamma,\epsilon}$, which is a path in V from p_0 to $s_v(p_0)$. The image $\pi(\sigma_{v,\gamma,\epsilon})$ is a loop in V/G based at $\bar{p_0}$. It isn't difficult to see when ϵ is small enough then the path $\sigma_{v,\gamma,\epsilon}$ is in M_G, and $\pi(\sigma_{v,\gamma,\epsilon})$ is in M_G/G. Moreover if ϵ is small enough the homotopy class of $\pi(\sigma_{v,\gamma,\epsilon})$ doesn't depend on ϵ, thus determine an element $s_{v,\gamma} \in \pi_1(M_G/G, \bar{p_0})$. We call this kind of elements in $\pi_1(M_G/G, \bar{p_0})$ as generators of monodromy.

Now suppose we have flat and G-invariant connection Ω, We denote the induced flat connection on the quotient bundle as $\bar{\Omega}$. By taking monodromy we obtain an operator $\psi : \pi_1(M_G/G, \bar{p_0}) \to A^{\times}$. Let $\rho : A \to End(E)$ be a representation of A. We have a flat and

G-invariant connection Ω_ρ on the bundle $E \times M_G$, which induce a flat connection $\bar{\Omega}_\rho$ on the quotient bundle $(E \times M_G)/G$. We have the monodromy representation $\psi_\rho : \pi_1(M_G, \bar{p_0}) \to GL(E)$. Relationship between these two kind of monnodromy representations can is simply:

$$\rho(\psi(\lambda)) = \psi_\rho(\lambda), for\ any\ \lambda \in \pi_1(M_G, \bar{p_0}).$$

Let $s_{\nu,\gamma} \in \pi_1(M_G, \bar{p_0})$ be a generator of monodromy. The element $\psi(s_{\nu,\gamma}) \in A$ and the morphism $\psi_\rho(s_{\nu,\gamma})$ can be described as follows. Let ϵ be small enough such that $\sigma_{\nu,\gamma,\epsilon}$ lies in M_G and it's homotopy class is $s_{\nu,\gamma}$. On the bundle of algebras $A \times M_G$, by parallel transportation along the path $\sigma_{\nu,\gamma,\epsilon}$, we obtain a morphism $T_{\gamma,\epsilon} : A_{p_0} \to A_{s_\nu(p_0)}$. Since when forming quotient A_{p_0} is identified with $A_{s_\nu(p_0)}$ by the correspondence $a \mapsto \phi(s_\nu)a$, so

$$\psi(s_{\nu,\gamma}) = \phi(s_\nu)^{-1} T_{\gamma,\epsilon}(1).$$

Similarly if $T_{\gamma,\epsilon,\rho} : E_{p_0} \to E_{s_\nu(p_\nu)}$ is parallel transportation of the connection Ω_ρ, we have

$$\psi_\rho(s_{\nu,\gamma}) = \rho(\phi(s_\nu))^{-1} T_{\gamma,\epsilon,\rho}.$$

3 Flat connections for BMW algebras

In this section we define a flat connection for BMW algebra $B_n(\tau, l)$. Using this connection we can deform any representation of the Brauer algebra $B_n(\tau)$ to be a one parameter family of representations of the braid group B_n, which will be shown to factor through $B_n(\tau, l)$.

We begin with some knowledge for hyperplane arrangements. Let E be a complex linear space. An hyperplane arrangement(or arrangement simply)in E simply means a finite set of hyperplanes contained in E .Arrangements arise in many fields of mathematics. For example, let (V, G) be a pseudo reflection group. Then the set of reflection hyperplanes of G is an arrangement. We denote it as $\mathcal{A}_G$. For an arrangement $\mathcal{A} = \{H_i\}_{i \in I}$ in E let $M_\mathcal{A} = E - \cup_{i \in I} H_i$ be the complementary space of all hyperplanes of $\mathcal{A}$. Topology and geometry of $M_\mathcal{A}$ are the main topics of arrangement theory.

Let $\mathcal{A} = \{H_i\}_{i \in I}$ be an arrangement in E. An edge of $\mathcal{A}$ is defined to be any nonempty intersection of elements of $\mathcal{A}$. If L is an edge of $\mathcal{A}$, define an subarrangement

$$\mathcal{A}_L = \{H_i \in \mathcal{A} | L \subset H_i\} = \{H_i\}_{i \in I_L}.$$

For every $i \in I$, chose a linear form f_i with kernel H_i. Define an 1 form on $M_\mathcal{A}$ as $\omega_i = d \log f_i$. Consider the formal connection $\Omega = \kappa \sum_{i \in I} X_i \omega_i$. Here X_i are linear operators to be determined. When we take X_i's as homeomorphisms of some linear space E, then Ω is realized as a connection on the bundle $M_\mathcal{A} \times E$. We have the following theorem.

Theorem 3.1. [19] *The formal connection Ω is flat if and only if the following conditions are satisfied.*

For any codimension 2 edge L of $\mathcal{A}$,and for any $i \in I_L$, $[X_i, \sum_{j \in I_L} X_j] = 0$.

Here $[A, B]$ means $AB - BA$.

Let E be a n-dimensional Euclidean space with an orthonormal basis $\{v_1, \cdots, v_n\}$. Suppose the corresponding coordinate system is $(x_1, \cdots, x_n)$. For $1 \le i < j \le n$, denote the

hyperplane $\ker(x_i - x_j)$ as $H_{i,j}$. Let $E^{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} < \nu_1, \cdots, \nu_n >$. Denote the corresponding coordinate system as $(z_1, \cdots, z_n)$. There is a natural imbedding $j : E \to E^{\mathbb{C}}$. Denote $\ker(z_i - z_j)$ as $H^{\mathbb{C}}_{i,j}$, which is called the complexification of $H_{i,j}$. We define $Y_n = E \setminus \cup_{i<j} H_{i,j}$, $Y^{\mathbb{C}}_n = E^{\mathbb{C}} \setminus \cup_{i<j} H^{\mathbb{C}}_{i,j}$. Define $\Delta = \{(x_1, \cdots, x_n) \in E | x_1 < x_2 < \cdots < x_n\}$ which is called a chamber. It isn't hard to see the walls of Δ are hyperplanes $\{H_{1,2}, H_{2,3}, \cdots, H_{n-1,n}\}$. Chose $p \in \Delta$. For $1 \le i \le n-1$, chose $p_i \in H_{i,i+1}$ lying in the closure of Δ and not lying in any other $H_{j,k}$. We chose path $\gamma_i : [0,1] \to E$ for any $1 \le i \le n-1$ satisfying the following conditions:

$$\gamma_i(0) = p, \ \gamma_i(1) = p_i, \ \gamma_i(t) \in \Delta \text{ for } 0 < t < 1.$$

The symmetric group S_n acts on $Y^{\mathbb{C}}_n$ freely by permuting the coordinates. Denote the quotient manifold $Y^{\mathbb{C}}_n / S_n$ as $X^{\mathbb{C}}_n$. Denote the quotient map from $Y^{\mathbb{C}}_n$ to $X^{\mathbb{C}}_n$ as π. We take $j(p)$, $j(\bar{p}) = \pi(j(p))$ as base point of $Y^{\mathbb{C}}_n$ and $X^{\mathbb{C}}_n$ respectively. According to the discussion in 2.2 , the paths $p \circ \gamma_i$ can be used to define a element $s_{\gamma_i} \in \pi_1(X^{\mathbb{C}}_n, j(\bar{p}))$ (generator of monodromy). The following result can be found in [6].

Proposition 3.1. [6] *The map* $\sigma_i \mapsto s_{\gamma_i}$ *extends to an isomorphism* $B_n \to \pi_1(X^{\mathbb{C}}_n, j(\bar{p}))$.

The following simple will be used later.

Lemma 3.1. *In the Brauer algebra* $B_n(\tau)$, *let* $s_{i,j} \in S_n \subset B_n(\tau)$ *be* (i,j) *permutation., let* $e_{i,j}$ *be as in introduction. we have*

(1) $e_{i,j} s_{k,l} = s_{k,l} e_{i,j}$ *if* $\{i,j\} \cap \{k,l\} = \varnothing$;

(2) $e_{i,j} e_{k,l} = e_{k,l} e_{i,j}$ *if* $\{i,j\} \cap \{k,l\} = \varnothing$;

(3) $e_{i,j} = e_{j,i}$;

(4) $e_{i,j} e_{i,k} = s_{j,k} e_{i,k} = e_{i,j} s_{j,k}$ *,for any different* i, j, k;

(5) $e^2_{i,j} = \tau e_{i,j}$ *, for any* $i \neq j$; *(6)* $s_{i,j} e_{j,k} = e_{i,k} s_{i,j}$.

Proof. They can be checked directly by using graghs.

For $1 \le i < j \le n-1$, define $\omega_{i,j} = d(z_i - z_j)/(z_i - z_j)$. Consider the formal connection $\bar{\Omega}_n = \kappa \sum_{i<j} (s_{i,j} - e_{i,j}) \omega_{i,j}$. We have

Proposition 3.2. *The formal connection* $\bar{\Omega}_n$ *is flat and* S_n *invariant.*

Proof. We certify $\bar{\Omega}_n$ satisfies conditions of theorem 3.1. For the arrangement $\mathcal{A}_n$,there are then following two type of codimension 2 edges

Case 1. $L = H_{i,j} \cap H_{k,l}$, $\{i,j\} \cap \{k,l\} = \varnothing$.

Whence $\mathcal{A}_L = \{H_{i,j}, H_{k,l}\}$. Now we have $s_{a,b} s_{c,d} = s_{c,d} s_{a,b}$ and $e_{a,b} e_{c,d} = e_{c,d} e_{a,b}$ if $\{a,b\} \cap \{c,d\} = \varnothing$. They are most easily seen by using graphs. so $[s_{i,j} - e_{i,j}, s_{k,l} - e_{k,l}] = 0$. Which gives

$$[s_{i,j} - e_{i,j}, s_{i,j} - e_{i,j} + s_{k,l} - e_{k,l}] = 0 = [s_{k,l} - e_{k,l}, s_{i,j} - e_{i,j} + s_{k,l} - e_{k,l}].$$

Case 2. $L = H_{i,j} \cap H_{j,k}$, where i, j, k are different. In this case $\mathcal{A}_L = \{H_{i,j}, H_{j,k}, H_{i,k}\}$,

$$\begin{aligned}
&[s_{i,j} - e_{i,j}, s_{i,k} - e_{i,k} + s_{j,k} - e_{j,k}] \\
=&[s_{i,j}, s_{i,k} + s_{j,k}] + (-e_{i,j}s_{i,k} + e_{i,j}e_{j,k}) + (s_{i,k}e_{i,j} - e_{j,k}e_{i,j}) \\
+&(-e_{i,j}s_{j,k} + e_{i,j}e_{i,k}) + (s_{j,k}e_{i,j} - e_{i,k}e_{i,j}) + [s_{i,j}, -e_{i,k} - e_{j,k}] \\
=&(-e_{i,j}s_{i,k} + e_{i,j}e_{j,k}) + (s_{i,k}e_{i,j} - e_{j,k}e_{i,j}) \\
+&(-e_{i,j}s_{j,k} + e_{i,j}e_{i,k}) + (s_{j,k}e_{i,j} - e_{i,k}e_{i,j}) + [s_{i,j}, -e_{i,k} - e_{j,k}] \\
=&(-e_{i,j}s_{i,k} + e_{i,j}e_{j,k}) + (s_{i,k}e_{i,j} - e_{j,k}e_{i,j}) + (-e_{i,j}s_{j,k} + e_{i,j}e_{i,k}) \\
+&(s_{j,k}e_{i,j} - e_{i,k}e_{i,j}) = 0.
\end{aligned}$$

The second equality is because $s_{i,j}s_{i,k} + s_{i,j}s_{j,k} = s_{j,k}s_{i,j} + s_{i,k}s_{i,j}$. For the third equality use Lemma 3.1, (6). For the fourth equality use lemma 3.1, (4). G-invariance of $\bar{\Omega}_n$ is evident.

Let (E, ρ) be a finite dimensional representation of $B_n(m)$. By proposition 3.1, the connection

$$\rho(\bar{\Omega}_n) = \kappa \sum_{i<j} (\rho(s_{i,j}) - \rho(e_{i,j}))\omega_{i,j}$$

induce a flat connection on the bundle $Y_n \times_{S_n} E$, which is a linear bundle on X_n. So by taking monodromy we obtain representations of the braid group B_n. We have

Theorem 3.2. *The monodromy representations of* B_n *constructed above factor through* $B_n(\tau, l)$*, for* $\tau = \frac{q^{1-m} - q^{m-1} + q^{-1} - q}{q^{-1} - q}$*,* $l = q^{m-1}$*. Where* $q = \exp \kappa\pi\sqrt{-1}$*.*

Proof. We first prove this theorem in case of $n = 2, 3$. The cases for $n \geq 4$ can be reduced to cases for $n = 2, 3$ by a result in local theory of meromorphic connections.

Case 1. $(n = 2)$ The Brauer algebra $B_2(m)$ is 3 dimensional, with a basis $\{1, s, e\}$, submit to relations $e^2 = me, se = es = e, s^2 = 1$. where 1 means the unit. Idempotents of this algebra are

$$\varepsilon_0 = \frac{e}{m}, \ \varepsilon_1 = \frac{1-s}{2}, \ \varepsilon_2 = \frac{1+s}{2} - \frac{e}{m}.$$

In this case the relevant reflection group is the symmetric group $(S_2, \mathbb{C})$, the only reflection hyperplane is $0 \in \mathbb{C}$. The complementary space is $\mathbb{C}^\times$. Let $p_0 = 1 \in \mathbb{C}^\times$ be a base point. Define

$$\sigma : [0, 1] \to \mathbb{C}^\times, t \mapsto e^{t\pi\sqrt{-1}}$$

which is a path from 1 to $-1 = s(1)$. Let $\pi : \mathbb{C}^\times \to \mathbb{C}^\times/S_2$ be the quotient map. Then the loop $\pi(\sigma)$ represent a generator of $\pi_1(\mathbb{C}^\times/S_2, \bar{p_0}) \cong \mathbb{Z}$, which is denoted as s_σ.

The flat connection is $\Omega = \kappa(s - e)dz/z$. Let F be a section of the bundle of algebras $\mathbb{C}^\times \times B_2(m)$, which can be presented as $F = F_0\varepsilon_0 + F_1\varepsilon_1 + F_2\varepsilon_2$. Where F_i's are functions on $\mathbb{C}^\times$. F is a flat connection when $dF = \Omega F$. It can be written as:

$$\frac{dF_0}{dz} = \frac{\kappa(1-m)}{z}F_0, \ \frac{dF_1}{dz} = \frac{-\kappa}{z}F_1, \ \frac{dF_2}{dz} = \frac{\kappa}{z}F_2.$$

The solution of this equation is $F(z) = z^{\kappa(1-m)}c_0\varepsilon_o + z^{-\kappa}c_1\varepsilon_1 + z^\kappa c_2\varepsilon_2$. where c_i are constants. Chose $1 \in \mathbb{C}^\times$ as the base point. If we request the initial condition $F(1) = 1$, then $c_0 = c_1 = c_2 = 1$. Through parallel transportation along the path σ, 1 is corresponded to the element $T_\sigma(1) = e^{\kappa(1-m)\pi\sqrt{-1}}\varepsilon_o + e^{-\kappa\pi\sqrt{-1}}\varepsilon_1 + e^{\kappa\pi\sqrt{-1}}\varepsilon_2$.

Let $\psi : \pi_1(\mathbb{C}^\times/S_2, \bar{p_0}) \to B_2(m)^\times$ be the monodromy morphism. We have

$$\psi(s_\sigma) = s^{-1}T_\sigma(1) = e^{\kappa(1-m)\pi\sqrt{-1}}\varepsilon_o - e^{-\kappa\pi\sqrt{-1}}\varepsilon_1 + e^{\kappa\pi\sqrt{-1}}\varepsilon_2.$$

Since ε_i's are idempotents, and $\sum_{i=0}^2 \varepsilon_i = 1$ and $\varepsilon_i\varepsilon_j = 0$ for $i \neq j$, so we have

$$(\psi(s_\sigma) - e^{\kappa(1-m)\pi\sqrt{-1}})(\psi(s_\sigma) + e^{-\kappa\pi\sqrt{-1}})(\psi(s_\sigma) - e^{\kappa\pi\sqrt{-1}}) = 0.$$

By definition the BMW algebra $B_2(\tau, l)$ is 3 dimensional with a basis $\{1, X, E\}$ submitting to the relations

$$XE = l^{-1}E = EX, \; E^2 = \tau E, \; l(X^2 + JX - 1) = JE.$$

where $J = \frac{l-l^{-1}}{1-\tau}$.

Let $q = e^{\kappa\pi\sqrt{-1}}$, $l = e^{\kappa(m-1)\pi\sqrt{-1}}$ and $\alpha = q^{-1} - q$, $\tau = \frac{l^{-1}-l+\alpha}{\alpha} = \frac{q^{1-m}-q^{m-1}+q^{-1}-q}{q^{-1}-q}$ From theorem 4.1, we see ψ factor through $\iota : \pi_1(\mathbb{C}^\times/S_2, \bar{p_0}) \to B_2(\tau, l)$ because if we set

$$X = \psi(s_\sigma), \; E = \frac{l}{q^{-1}-q}(\psi(s_\sigma)^2 + (q^{-1} - q)\psi(s_\sigma) - 1),$$

then all relations for BMW algebras are satisfied.

Case 2. $n = 3$. We only consider those $B_3(m)$'s which are semisimple. We prove for any finite dimensional representation $\rho : B_3(m) \to End(E)$, the monodromy representation of $\rho(\bar{\Omega}_3) : B_3 \to Aut(E)$ factor through $\iota : B_3 \to B_3(\tau, l)$, where τ, l are defined as in case 1. For this aim we only need to consider the cases when ρ is an irreducible representation. It isn't hard to see that $B_3(m)$ has 4 indifferent irreducible representations as follows:

ρ_1: $E = \mathbb{C} < v >$. $s_i v = v$ for any i; $e_i v = 0$ for any i.

ρ_2: $E = \mathbb{C} < v >$. $s_i v = -v$ for any i; $e_i v = 0$ for any i.

ρ_3: $E = \mathbb{C} < v_1, v_2 >$. $e_i v_j = 0$ for any i, j. The actions of s_i's make E becoming the canonical representation of S_3.

ρ_4: $E = \mathbb{C} < v_{1,2}, v_{2,3}, v_{1,3} >$. $s_{i,j}v_{i,j} = v_{i,j}$. For $i > j$, we set $v_{i,j} = v_{j,i}$ and $s_{i,j} = s_{j,i}$. For different i, j, k, $s_{i,j}v_{j,k} = v_{i,k}$. $e_{i,j}v_{i,j} = m v_{i,j}$. For different i, j, k, $e_{i,j}v_{j,k} = v_{i,j}$.

In the case of ρ_i, let $\psi_i : \pi_1(X_n^{\mathbb{C}}, j(\bar{p})) \to Aut(E)$ be the monodromy representation of the connection $\rho_i(\bar{\Omega}_3)$. For $k = 1, 2$, we set $X_k = \psi_i(s_{\gamma_k})$ and $E_k = \frac{l}{q^{-1}-q}(\psi(s_{\sigma_k})^2 + (q^{-1} - q)\psi(s_{\sigma_k}) - 1)$.

In the case of ρ_1, ρ_2, ρ_3, the elements e_i's act as 0, so the connection $\rho_i(\bar{\Omega}_3)$ in fact degenerates to the flat connection for Hecke algebras. It is readily to check in these cases $\psi(s_{\sigma_k})^2 + (q^{-1} - q)\psi(s_{\sigma_k}) - 1 = 0$. So $E_k = 0$ in these cases.

In the case of ρ_4, we notice this is nothing but the infinisimal Krammer representation defined by Marin. This case is proved by using theorem 4.1 in [21].

4 Generalized Lawrence-Krammer Representations

Let V be a n-dimensional complex linear space. Let $G \subset GL(V)$ be a finite reflection group. Let R be the set of reflections in G. For each $s \in R$, denote the reflection hyperplane of s,namely the subspace $Ker(s-1)$ as H_s. Let $M_G = V \setminus \bigcup_{s \in R} H_s$ be the complementary space of all reflection hyperplanes.

The generalized LK representations of B_G of Marin are described by certain flat connections as follows. First, for every H_s, let α_s be a nonzero linear form with kernel H_s. Let

$\omega_s = d\alpha_s/\alpha_s$, which is a closed,holomorphic 1-form on M_G. Then let $V_G = \mathbb{C}\langle v_s\rangle_{s\in R}$ be a complex linear space with a basis indexed by R. For every pair of elements $s, u \in R$, define a nonnegative integer $\alpha(s,u) = \#\{r \in R | rur = s\}$. Chose a constant $m \in \mathbb{C}$. For any $s \in R$, define a linear operator $t_s \in GL(V_G)$ as follows:

$$t_s \cdot v_s = m v_s, \quad t_s \cdot v_u = v_{sus} - \alpha(s,u) v_s \quad \text{for} \quad s \neq u.$$

Chose another constant $k \in \mathbb{C}$. Define a connection $\Omega_K = \sum_{s\in R} k \cdot t_s \omega_s$ on the trivial bundle $V_G \times M_G$.

Theorem and Definition ([22]) The connection Ω_K is flat and G-invariant. So it induce a flat connection $\bar{\Omega}_K$ on the quotient bundle $(V_G \times M_G)/G$. The generalized LK representation for B_G is defined as the monodromy representation of $\bar{\Omega}_K$.

We denote the generalized Krammer representation as $(V, \rho_{\kappa,m})$. When (G, V) is the reflection group W_Γ of ADE type, they were first constructed in [9] by Cohen, Wales and by Digne in [13]. They are proved to factor through BMW algebras in [10].

Theorem 4.1. [21] *The generalized Krammer representation* $(V, \rho_{\kappa,m})$*factor through the generalized BMW algebra* $B_\Gamma(\tau, l)$ *with* $\tau = \frac{q^m - q^{-m} + q^{-1} - q}{q^{-1} - q}$ *and* $l = q^{-m}$. *Where* $q = e^{\kappa\pi\sqrt{-1}}$.

For later convenience we change notations slightly. For $s \in \mathcal{R}$, we define $p_s : V_G \to V_G$ by

$$p_s(v_s) = (1 - m_s)v_s, \; p_s(v_u) = \alpha(s,u)v_s \text{ for } u \neq s.$$

We also define $\iota : G \to Aut(V_G)$ by $\iota(w)(v_s) = v_{wuw^{-1}}$. Then p_s is a projector to the complex line $\mathbb{C}v_s$. And Marin's flat connection Ω_K is written as $\sum_{s\in\mathcal{R}} k \cdot (\iota(s) - p_s)\omega_s$.

Now we let V be a n-dimensional complex linear space. Now let $G \subset GL(V)$ be a finite pseudo reflection group(not only reflection group). Let R be the set of pseudo reflections in G. For $s \in R$, let H_s be the hyperplane fixed by s. Let $\mathcal{A} = \{H_i\}_{i\in P}$ be the arrangement of reflection hyperplanes of G. Define $V_G = \mathbb{C} < v_i >_{i\in P}$. Since $w(H_v)$ is annother reflection hyperplane for any $w \in G$ and $v \in P$, there is an action of G on P which induce a representation $\iota : G \to Aut(V_G)$. We also define $w(i)$ by $H_{w(i)} = w(H_v)$.

For $i \in P$, let $p_i : V_G \to V_G$ be a projector to $\mathbb{C}v_i$ which is written as:

$$p_i(v_i) = m_i v_i, \; p_i(v_j) = \alpha_{i,j} v_i.$$

As in Section 2.2 let $\{\mu_s\}_{s\in R}$ be a set of constants such that: $\mu_{s_1} = \mu_{s_2}$ if s_1 is conjugate to s_2 in G. Define a function $\nu : R \to P$ such that $H_{\nu(s)}$ is the reflection hyperplane of s for any $s \in R$.

Consider a connection Ω_{LK} on the trivial bundle $V_G \times M_G$ which have the form

$$\kappa \sum_{\nu\in P} \Big(\sum_{s:\nu(s)=\nu} \mu_s \iota(s) - p_\nu \Big) \omega_\nu.$$

The following theorem generalize Marin's construction to psuedo reflection groups and with more parameters in some cases.

Theorem 4.2. *The connection* Ω_{LK} *is flat and* G *invariant if and only if the the following conditions are satisfied:*
(1) $m_i = m_j$ *if there is* $w \in G$ *such that* $\iota(w)(v_i) = v_j$.
(2) $\alpha_{i,j} = \sum_{s:\iota(s)(v_j)=v_i} \mu_s$.

Proof. Suppose Ω_{LK} is a flat, G invariant connection. Let $w \in G$,

$$w^*(\Omega_{LK}) = \sum_{i\in I}(\sum_{s:i(s)=i} \mu_s \iota(w)\iota(s)\iota(w)^{-1} + \iota_w p_i \iota_w^{-1})\omega_{w(i)}.$$

The connection Ω_{LK} being G invariant means: for any $w \in G$, $w^*(\Omega_{LK}) = \Omega_{LK}$. Because $\{\omega_i\}_{i\in I}$ constitute a basis of $H^1_{DR}(M_G, \mathbb{C})$, so it is equivalent to $\iota_w p_i \iota_w^{-1} = p_{w(i)}$, which is equivalent to: $m_i = m_{w(i)}$.

Let L be any codimension 2 edge of the arrangement $\mathcal{A}$. Let $H_{i_1}, \cdots, H_{i_N}$ be all the hyperplanes in $\mathcal{A}$ containing L. The condition of Ω being flat is written as :

$$[\sum_{s:i(s)=i_a} \mu_s \rho(s) - p_{i_a}, \sum_{\nu=1}^{N}(\sum_{s:i(s)=i_\nu} \mu_s \rho(s) - p_{i_\nu})] = 0.$$

for $1 \le a \le N$. We discuss the case $a = 1$ first. Since Ω_0 is a flat connection, it is equivalent to:

$$[p_{i_1}, \sum_{\nu=1}^{N}(\sum_{s:i(s)=i_\nu} \mu_s \iota(s) p_{i_\nu})] + [\sum_{s:i(s)=i_1} \mu_s \iota(s), \sum_{\nu=1}^{N} p_{i_\nu})] = 0.$$

Now for those s such that $i(s) = i_1$ we have $\{s(i_1), \cdots, s(i_N)\} = \{i_1, \cdots, i_N\}$. So we have:

$$\begin{aligned}
[\sum_{s:i(s)=i_1} \mu_s \iota(s), \sum_{\nu=1}^{N} p_{i_\nu})] &= \sum_{s:i(s)=i} \mu_s \sum_{\nu=1}^{N}(\iota(s)p_{i_\nu} - p_{i_\nu}\iota(s)) \\
&= \sum_{s:i(s)=i} \mu_s \sum_{\nu=1}^{N}(p_{s(i_\nu)}\iota(s) - p_{i_\nu}\iota(s)) \\
&= 0.
\end{aligned} \tag{1}$$

So the identity (2) is equivalent to:

$$\begin{aligned}
[p_{i_1}, \sum_{\nu=1}^{N}(\sum_{s:i(s)=i_\nu} \mu_s \iota(s) + p_{i_\nu})] &= \sum_{\nu=2}^{N} \sum_{s:i(s)=i_\nu} [p_{i_1}, (\sum_{s:i(s)=i_\nu} \mu_s \iota(s) + p_{i_\nu})] \\
&= 0.
\end{aligned} \tag{2}$$

This is because $[p_{i_1}, s] = 0$ if $i(s) = i_1$. After splitting the Lie bracket in equation (2), the sum of all those terms mapping to $\mathbb{C}v_{i_u}$ is $p_{i_u}p_{i_1} - \sum_{s:s(v_{i_1})=v_{i_u}} \mu_s \iota(s) p_{i_1}$. It must be 0. Chose s_0 such that $s_0(v_{i_1}) = v_{i_u}$, then e have $p_{i_u}p_{i_1} = \alpha_{i_u,i_1}\iota(s_0)p_{i_1}$. More over, for any s such that $s(v_{i_1}) = v_{i_u}$ we have $\iota(s)p_{i_1} = \iota(s_0)p_{i_1}$. Put these identities in equation (2), we get

$$(\alpha_{i_u,i_1} - \sum_{s:s(v_{i_1})=v_{i_u}} \mu_s)\iota(s_0)p_{i_1} = 0.$$

So we have $\alpha_{i_u,i_1} = \sum_{s:s(v_{i_1})=v_{i_u}} \mu_s$.

Now suppose conditions is satisfied, by the same arguments we only need to prove above equation (4) to show Ω is flat. The conditions (2) implies

$$p_i p_j = \sum_{s:s(j)=i} \mu_s \iota(s) p_j, \text{for any } i \neq j. \tag{3}$$

It also implies

$$p_i p_j = \sum_{s:s(j)=i} \mu_s p_i \iota(s), \text{for any } i \neq j. \tag{4}$$

since $\iota(s)p_j = \iota(s)p_j\iota(s)^{-1}\iota(s) = p_i\iota(s)$ for those s such that $s(j) = i$. Now the right hand side of equation (4) can be written as

$$\sum_{\nu=2}^{N}(p_{i_1}p_{i_\nu} - \sum_{s:s(i_\nu)=i_1} \mu_s p_{i_1}\iota(s)) + \sum_{\nu=2}^{N}(p_{i_\nu}p_{i_1} - \sum_{s:s(i_\nu)=i_1} \mu_s \iota(s) p_{i_\nu}).$$

So the equation (2) is true by using identities (3),(4). The G invariance of the connection isn't hard to see.

□

Remark In the connection Ω_{LK} if make $\mu_s = \kappa$ for all s and $m_i = m$ for all i then we obtain Marin's connection. Above theorem give another proof that the connection is flat. It explains why the numbers $\alpha_{i,j}$ will appear in Marin's construction, also show the LK representation is unique in some sense. We can define and describe the generalized LK representation as follows.

Definition 4.1 (Generalized LK representation for all complex braid groups). *Following notations introduced above. Since Ω_{LK} is G-invariant, it induces a flat connection $\bar{\Omega}_{LK}$ on the quotient bundle $V_G \times_G M_G$. the generalized Krammer representation of the braid group A_G is defined as the monodromy representation of $\bar{\Omega}_{LK}$.*

5 Generalization of Brauer Algebras

5.1 Definition and Main Properties

Let (V, G) be any finite pseudo reflection group. Denote the set of pseudo reflections in G as R, and let $\mathcal{A} = \{H_i\}_{i\in P}$ be the set of reflection hyperplanes. For $i, j \in P$, Let $R(i, j) = \{s \in R \mid s(H_j) = H_i\}$. For $i \in P$, as in 2.2 let G_i be the subgroup of G consisting of elements that fixing all points in H_i, let $m_i = |G_i|$, and let s_i be the unique element in G_i with exceptional eigenvalue $e^{\frac{2\pi\sqrt{-1}}{m_i}}$. we write $s_1 \sim s_2$ for $s_1, s_2 \in R$ if they are in the same conjugacy class, and write $i \sim j$ for $i, j \in P$ if $w(i) = j$ for some $w \in G$. Chose $\mu_s \in \mathbb{C}$ for every $s \in R$ and $m_i \in \mathbb{C}$ for every $i \in P$ such that

$$\mu_{s_1} = \mu_{s_2} \text{ if } s_1 \sim s_2, m_i = m_j \text{ if } i \sim j.$$

The data $\{\mu_s, m_i\}_{s\in\mathcal{R}, i\in P}$ will be denoted by one symbol Υ. We define an algebra $B_G(\Upsilon)$ as follows.

Definition 5.1. *The algebra $B_G(\Upsilon)$ associated with pseudo reflection group (V, G) is generated by $\{\bar{w}\}_{w\in G} \cup \{e_i\}_{i\in P}$, submitting to the following relations.*

(0) $\bar{w}_1\bar{w}_2 = \bar{w}_3$ *if* $w_1w_2 = w_3$;

(1) $\bar{s}_i e_i = e_i\bar{s}_i = e_i$, *for* $i \in P$; $s_i \in G$ *is any pseudo reflection with reflection hyperplane* H_i.

(2) $e_i^2 = m_i e_i$ *;*

(3) $\bar{w}e_j = e_i\bar{w}$ *, if* $w \in G$ *satisfies* $w(H_j) = H_i$*;*

(4) $e_ie_j = e_je_i$*, if* $\{\ k \in P|\ H_k \supset H_i \cap H_j\} = \{i, j\}$ *;*

(5) $e_ie_j = (\sum_{s\in R(i,j)} \mu_s s)e_j$ *, if* $\{k \in P|H_k \supset H_i \cap H_j\} \neq \{i, j\}$*, and* $R(i, j) \neq \emptyset$*.*

(6) $e_ie_j = 0$*, if* $\{k \in I|H_k \supset H_i \cap H_j\} \neq \{i, j\}$ *and* $R(i, j) = \emptyset$*.*

Remark 5.1. *Relation* (0) *is nothing but letting* $\mathbb{C}G$ *be imbedded in* $B_G(\Upsilon)$*. Relation* (3) *along with relation* (5) *implies* $e_ie_j = e_i(\sum_{s\in R(i,j)} \mu_s s)$*.*

When (G, V) *is a complex reflection group, there is a bijection from* R *to* $\mathcal{A}$*:* $s \mapsto H_s$*. So we can use* R *as the indices set of reflection hyperplanes, and* $\mathcal{A} = \{H_s\}_{s\in R}$*. In these cases, for* $s_1, s_2 \in R$*,* $R(s_1, s_2) = \{s \in R|s(H_{s_2}) = H_{s_1}\} = \{s \in R|ss_2s = s_1\}$*.*

When G *is a infinite Coxeter group, by using geometric representation of* G *above definition still make sense.* $B_G(\Upsilon)$ *for those cases can also be defined by a canonical presentation as shown in section 5.3.*

First we have

Theorem 5.1. $B_G(\Upsilon)$ *is a finite dimensional algebra. And* $w \mapsto \bar{w}$ *for* $w \in G$ *induce an injection* $j : \mathbb{C}G \to B_G(\Upsilon)$*.*

Proof. First by using relation (3), we can identify any word made from the set $\{w \in G\}\coprod\{e_i\}_{i\in P}$ to a word of the form $we_{i_1}e_{i_2}\cdots e_{i_k}$ where $w \in G$. We call $e_{i_1}e_{i_2}\cdots e_{i_k}$ as the 'e-tail' of the word $we_{i_1}e_{i_2}\cdots e_{i_k}$. Then if two neighboring e_{i_ν} and $e_{i_{\nu+1}}$ don't commute with each other, then conditions in (5) are satisfied as can be seen in the next lemma.

Lemma 5.1. *If two pseudo reflection* s_1 *and* s_2 *don't commute with each other, suppose the reflection hyperplane of* $s_1(s_2)$ *is* $H_{i_1}(H_{i_2})$*, then* $\{i_1, i_2\} \subsetneq \{k \in P|H_k \supseteq H_{i_1} \cap H_{i_2}\}$*.*

Proof. We suppose $\{i_1, i_2\} = \{k \in P|H_k \supseteq H_{i_1} \cap H_{i_2}\}$. Let $L = H_{i_1} \cap H_{i_2}$, and $< , >$ being a G-invariant inner product on V. Chose $v_k \in H_{i_k}$ such that $v_k \perp L$ according to $< , >$ for $k = 1, 2$. Suppose $\{v_3, \cdots , v_N\}$ is a basis of L, then $\{v_1, v_2, \cdots , v_N\}$ is a basis of V. Now since $s_1(H_{i_2})$ is another reflection hyperplane containing L and $s_1(H_{i_2}) \neq H_{i_1}$, so we have $s_1(H_{i_2}) = H_{i_2}$, which implies s_1 can be presented as a diagonal matrix according to the basis $\{v_1, \cdots , v_N\}$. Similarly s_2 can be presented by a diagonal matrix according to the same basis. So $s_1s_2 = s_2s_1$ which is a contradiction. ☐

The first statement follows from the next lemma.

Lemma 5.2. *The algebra* $B_G(\Upsilon)$ *is spanned by the set*

$$\{w \in G_m\}\coprod\{we_{i_1}\cdots e_{i_M}|w \in G,\ e_{i_u}e_{i_\nu} = e_{i_\nu}e_{i_u};\ i_\nu \neq i_u \text{ if } u \neq \nu\ ; M \geq 1\}$$

Proof. Let A be the space in $B_G(\Upsilon)$ spanned by elements listed in the lemma. By applying (2) in definition 5.1, we see the algebra is spanned by elements having the form $x = we_{i_1}\cdots e_{i_K}$ where $w \in G_m$. For convenience we call $e_{i_1}\cdots e_{i_K}$ as the e-tail of the word x, and K as the length of it's e-tail. We prove that every such element belongs to A by induction on the length of their e-tails K. First this is true if $K = 1$. Suppose it is true for

$K \leq M$. Now suppose $x = we_{i_1} \cdots e_{i_K}$ such that $K = M + 1$. If there are two neighboring e_{i_ν}, $e_{i_{\nu+1}}$ don't commute, then lemma 5.1 enable us to apply (5) or (6) in definition 5.1 to identify x with a linear sum of words whose e-tail length are smaller than $M + 1$. Suppose all e_{i_ν}'s in x commute with each other, if there are ν_1, ν_2 such that $i_{\nu_1} = i_{\nu_2}$, we use transpositions between e_{i_ν}'s to identify x with a word $y = we_{j_1} \cdots e_{j_{M+1}}$ such that $j_1 = j_2$. So $x = y = m_{j_1} we_{j_2} \cdots e_{j_{M+1}}$. If all e_{i_ν}'s commute and all i_ν's are different then $x \in A$, and induction is completed.

For the second statement, it isn't hard to see the following map

$$s \mapsto s, \text{ for } s \in R; \; e_i \mapsto 0, \text{ for } i \in P$$

extends to a surjection $\pi : B_G(\Upsilon) \to \mathbb{C}G$, and $\pi \circ j = id$. So j is injective. □

This completes the proof of Theorem 5.1. □

By Theorem 5.1, $\mathbb{C}G$ is naturally embedded in $B_G(\Upsilon)$. From now on we will denote $\bar{w}$ in $B_G(\Upsilon)$ simply as w.

The next lemma reduce one parameter in $B_G(\Upsilon)$.

Lemma 5.3. *For $\lambda \in \mathbb{C}^\times$, Let $\mu'_s = \lambda\mu_s$ for $s \in R$, and Let $m'_i = \lambda m_i$ for $i \in P$. Let $\Upsilon' = \{\mu'_s, m'_i\}_{s\in R, i\in P}$, then $B_G(\Upsilon') \cong B_G(\Upsilon)$.*

Proof. Denote the generators of $B_G(\Upsilon')$ appeared in Definition 5.1 as S_α's and E_i's . Then

$$S_\alpha \mapsto s_\alpha, \; E_i \mapsto e_i \text{ for } \alpha \in \Phi, \text{ and } i \in P$$

extend to an isomorphism from $B_G(\Upsilon')$ to $B_G(\Upsilon)$.

□

The following lemma can be found in [22].

Lemma 5.4. *For two different hyperplane $H_i, H_j \in \mathcal{A}$, If $s \in R$ satisfies $s(H_j) = H_i$, then s fix all points in $H_i \cap H_j$.*

Proof. Let $<\ ,\ >$ be a G invariant, positive definite Hermitian form on V. Let ϵ be the exceptional eigenvalue of s, and let α be an eigenvalue of s with eigenvalue ϵ. Let α_i, α_j be some nonzero vector perpendicular to H_i, H_j respectively. Then $\alpha \perp H_s$. The action of s can be written as $s(\nu) = \nu - (1-\epsilon)\frac{<\nu,\alpha>}{<\alpha,\alpha>}\alpha$. Now $s(H_j) = H_i$ implies $s(\alpha_j) = \alpha_j - (1-\epsilon)\frac{<\alpha_j,\alpha>}{<\alpha,\alpha>}\alpha = \lambda\alpha_i$ for some $\lambda \neq 0$. Denote $(1-\epsilon)\frac{<\alpha_j,\alpha>}{<\alpha,\alpha>}$ as κ. The condition that H_i is different from H_j implies $\kappa \neq 0$. So we have $\alpha = \frac{1}{\kappa}(\alpha_j - \lambda\alpha_i) \perp H_i \cap H_j$, and s fix all points in $H_i \cap H_j$.

□

There is a natural anti-involution on $B_G(\Upsilon)$ which may be used to construct a cellular structure.

Lemma 5.5. *The correspondence*

$$w \mapsto w^{-1} \text{ for } w \in G \subset B_G(\Upsilon), \; e_i \mapsto e_i \text{ for all } i \in P$$

extends to an anti-involution $*$ of $B_G(\Upsilon)$ if $\mu_s = \mu_{s^{-1}}$ for any $s \in R$.

Proof. We only need to certify $*$ keep all relations in definition 5.1. As an example for relation (5), on the one hand $*(e_ie_j) = e_je_i$, on the other hand $*[(\sum_{s\in R(i,j)} \mu_s s)e_j] = e_j(\sum_{s\in R(i,j)} \mu_s s^{-1}) = (\sum_{s\in R(i,j)} \mu_s s^{-1})e_i = (\sum_{s\in R(j,i)} \mu_{s^{-1}} s)e_i = (\sum_{s\in R(j,i)} \mu_s s)e_i$. $\square$

Next we construct some connections for $B_G(\Upsilon)$. Suppose $\rho : B_G(\Upsilon) \to End(E)$ is a finite dimensional representation. On the vector bundle $E \times M_G$, we define a connection

$$\Omega_\rho = \kappa \sum_{i\in P}(\sum_{s:i(s)=i} \mu_s\rho(s) - \rho(e_i))\omega_i$$

where $\kappa \in \mathbb{C}$. Let G acts on $E \times M_G$ as $w \cdot (v, x) = (\rho(w)v, wx)$ for $w \in G$ and $(v, x) \in E \times M_G$. Then we have

Proposition 5.1. *The connection Ω_ρ is flat and G-invariant.*

Proof. It is enough to deal with the case $\kappa = 1$. The connection Ω_ρ is G-invariant if and only if for any $w \in G$,

$$\sum_{i\in P}(\sum_{s:i(s)=i} \mu_s\rho(w)\rho(s)\rho(w)^{-1} - \rho(w)\rho(e_i)\rho(w)^{-1})\omega_{w(i)} = \sum_{i\in P}(\sum_{s:i(s)=i} \mu_s\rho(s) - \rho(e_i))\omega_i. \quad (5)$$

By definition 5.1 (3), we have $\rho(w)\rho(e_i)\rho(w)^{-1} = \rho(we_iw^{-1}) = \rho(e_{w(i)})$. And $\{wsw^{-1}|i(s) = i\} = \{s|i(s) = w(i)\}$, so identity (7) follows.

Let L be any codimension 2 edge for the arrangement $\mathcal{A}$, and let $H_{i_1}, \cdots, H_{i_N}$ be all the hyperplanes in $\mathcal{A}$ containing L. By theorem 3.1, to prove Ω_ρ is flat we need to show for any u

$$[\sum_{s:i(s)=i_u} \mu_s\rho(s) - \rho(e_{i_u}), \sum_{v=1}^{N}(\sum_{s:i(s)=i_v} \mu_s\rho(s) - \rho(e_{i_v}))] = 0. \quad (6)$$

Now remember the connection $\kappa \sum_{i\in P}(\sum_{s:i(s)=i} \mu_s\rho(s)\omega_i)$ is flat by proposition 2.1, so (8) is equivalent to

$$-[\sum_{s:i(s)=i_u} \mu_s\rho(s), \sum_{v=1}^{N} \rho(e_{i_v})] - [\rho(e_{i_u}), \sum_{v=1}^{N}(\sum_{s:i(s)=i_v} \mu_s\rho(s) - \rho(e_{i_v}))] = 0 \quad (7)$$

Because for any s such that $i(s) = i_u$, there is $\{s(H_{i_1}), \cdots, s(H_{i_n})\} = \{H_{i_1}, \cdots, H_{i_N}\}$, so

$$\begin{aligned}\rho(s)\sum_{v=1}^{N} \rho(e_{iv}) - \sum_{v=1}^{N} \rho(e_{iv})\rho(s) &= (\rho(s)\sum_{v=1}^{N} \rho(e_{iv})\rho(s)^{-1} - \sum_{v=1}^{N} \rho(e_{iv}))\rho(s) \\ &= (\sum_{v=1}^{N} \rho(e_{s(i_v)}) - \sum_{i=1}^{N} e_{i_v}) = 0\end{aligned} \quad (8)$$

So (9) is equivalent to

$$[\rho(e_{i_u}), \sum_{v=1}^{N}(\sum_{s:i(s)=i_v} \mu_s\rho(s) - \rho(e_{i_v}))] = 0. \quad (9)$$

We define $I_1 = \{1 \le v \le N | s(i_v) = i_u, \text{for some } s \in \mathcal{R}\}$, and $I_2 = \{1 \le v \le N | s(i_v) \ne i_u, \text{for any } s \in \mathcal{R}\}$. There is $\{1, 2, \cdots, N\} = I_1 \coprod I_2$.

$$
\begin{aligned}
[\rho(e_{i_u}), \sum_{\nu=1}^{N}(\sum_{s:i(s)=i_\nu} \mu_s\rho(s) - \rho(e_{i_\nu}))] &= -\sum_{\nu\in I_1}(e_{i_u}e_{i_\nu} - \sum_{s:s(i_\nu)=i_u} \mu_s e_{i_u}\rho(s)) \\
&+ \sum_{\nu\in I_1}(e_{i_\nu}e_{i_u} - \sum_{s:s(i_u)=i_\nu} \mu_s\rho(s)e_{i_u}) \\
&+ \sum_{\nu\in I_2}(e_{i_u}e_{i_\nu} - e_{i_\nu}e_{i_u}) \\
&= 0.
\end{aligned} \tag{10}
$$

Where we used relation (5),(6),(8) in definition 5.1.

□

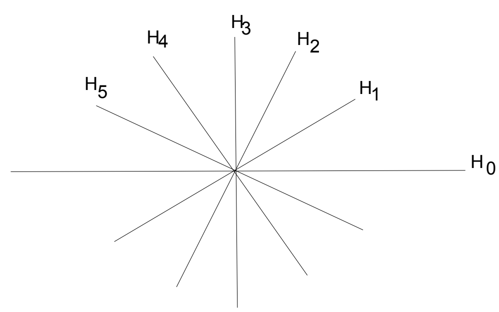
Remark 5.2. *Using notations in section 4, it is direct to check the map $w \mapsto \iota(w)$, $e_i \mapsto p_i$ define a representation $B_G(\Upsilon) \to gl(V_G)$. So from $B_G(\Upsilon)$ we can obtain the generalized Lawrence-Krammer representation.*

5.2 Low rank cases

In this subsection we study the algebra $\mathcal{B}_G(\Upsilon)$ when G is a dihedral group, namely a reflection group of type $I_2(m)(m \geq 2)$. These cases are relatively simple but fundamental. We prove $\mathcal{B}_G(\Upsilon)$ has a cellular structure and be semisimple for generic parameters.

Denote the dihedral group of type $I_2(m)$ as G_m.

The arrangement of its reflection hyperplanes can be explained with the following graph.



m=6

There are m lines(hyperplanes) passing the origin. The angle between every two neighboring line is π/m. Suppose the x-axis is one of the reflection lines and denote it as H_0, we denote these lines by $H_0, H_1, \cdots, H_{m-1}$ in anti-clockwise order as shown in above graph. Denote the reflection by H_i as s_i. The set of reflections in G_m is $R = \{s_i\}_{0\leq i\leq m-1}$. It is well known that G_m is generated by s_0, s_1 with the following presentation

$$
< s_0\ , s_1,\ |(s_0s_1)^m = 1, s_0^2 = s_1^2 = 1 > .
$$

Under this presentation, s_i can be determined inductively in the following way. $s_1 = s_1$, $s_2 = s_1s_0s_1$ and $s_i = s_{i-1}s_{i-2}s_{i-1}$. By $[s_is_j\cdots]_k$ we denote the unique word starting with s_is_j, in which s_i and s_j appear alternatively, and whose length is k. The word $[\cdots s_0s_1]_k$ is defined similarly. Then $s_i = [s_1s_0\cdots]_{2i-1}$, and $s_m = [s_1s_0\cdots]_{2m-1} = s_0$.

For $k \in \mathbb{Z}$, let $[k]$ be the unique number in $\{0, 1, \cdots, m-1\}$ such that $k \equiv [k] \mathrm{mod}(m\mathbb{Z})$. It turns out the structure of $B_{G_m}(\Upsilon)$ when m is odd is rather different from when m is even.

$B_{G_m}(\Upsilon)$ with m being odd. For $0 \leq j < i \leq m-1$, a number $k(i,j)$ is defined as follows. When $i-j$ is even, let $k(i,j) = (i+j)/2$; when $i-j$ is odd, let $k(i,j) = [(m+i+j)/2]$.

Then we have $R(i,j) = \{k(i,j)\}$ for any $0 \leq j < i \leq m-1$.

For $s_i \in R$, we denote μ_{s_i}, τ_{s_i} by μ_i, τ_i simply. Since all reflections lie in the same conjugacy class, so all μ_i's equal some μ and all τ_i's equal some τ. For later use we translate Definition 5.1 in these cases as follows.

Definition 5.2. *The algebra* $B_{G_m}(\Upsilon)$ *is generated by* $s_0, s_1;\ e_0, e_1, \cdots, e_{m-1}$ *with relations:*

(1) $s_0^2 = s_1^2 = 1$.

(2) $(s_0 s_1)^m = 1$.

(3) $s_i e_i = e_i s_i = e_i$, for $0 \leq i \leq m-1$. *Where* $s_i = [s_1 s_0 \cdots]_{2i-1}$.

(4) $we_i = e_j w$, *for a word* w *composed by* s_0, s_1 *such that* $ws_i = s_j w$ *is an consequence of relations* (1) *and* (2).

(5) $e_i^2 = \tau e_i$, for $0 \leq i \leq m-1$.

(6) $e_i e_j = s_{k(i,j)} e_j = e_i s_{k(i,j)}$ *for any* $i \neq j$.

As lemma 5.2 shows, the algebra $B_{G_m}(\Upsilon)$ is spanned by the set $T_m = G_m \coprod \{we_i | w \in G_m, 0 \leq i \leq m-1\}$. Because of above relation (3), $|\{we_i | w \in G_m\}|$ doesn't exceed the number of left cosets of $\{1, s_i\}$ in G_m which equal m. So

$$\mathrm{dim} B_{G_m}(\Upsilon) \leq |T_m| \leq 2m + m^2.$$

We want to show $\mathrm{dim}\mathcal{B}_{G_m}(\Upsilon) \geq 2m + m^2$. Let $\pi : B_{G_m}(\Upsilon) \to \mathbb{C}G_m$ be the natural projection. Let (V_i, ρ_i) $(1 \leq i \leq K)$ be all irreducible representations of G_m. Through π, every (V_i, ρ_i) induce a irreducible representation of $B_{G_m}(\Upsilon)$ which is denoted by $(V_i, \bar{\rho}_i)$. Then obviously $\bar{\rho}_i$ isn't equivalent to $\bar{\rho}_j$ if $i \neq j$. Beside of these induced representations, $B_{G_m}(\Upsilon)$ has one more representation coming from Marin's generalized LK representations. Let V_{G_m} be an vector space with basis $\{v_0, \cdots, v_{m-1}\}$. Define a representation (V_{G_m}, ρ) of G_m by setting $\rho(w)(v_i) = v_{w(i)}$, where $w(i)$ is determined by $ws_i w^{-1} = s_{w(i)}$. For every i, define a projection $p_i \in \mathrm{End}(V_{G_m})$ as

$$p_i(v_i) = \tau v_i;\ p_i(v_j) = \alpha_{i,j} v_i \text{ for } j \neq i,$$

where $\alpha_{i,j} = \#\{k | s_k s_j s_k = s_i\}$. In present cases $\alpha_{i,j} = 1$ for any $i \neq j$.

Lemma 5.6. *The map* $w \mapsto \rho(w)$, *for* $w \in G_m$, $e_i \mapsto p_i$ *extends to a representation* $\rho_{Kr} : B_{G_m}(\Upsilon) \to \mathrm{End}(V_{G_m})$.

Proof. This can be proved by directly checking relations. □

It isn't hard to see that every v_i is a generator of the representation (V_{G_m}, ρ_{Kr}). Now for $0 \neq v = \sum a_i v_i \in V_{G_n}$, $p_i(v) = (\sum_{j \neq i} a_j + \tau a_i)v_i$. We see if $\tau \neq 1$, there exists some i such that $p_i(v) \neq 0$. So we have proved

Lemma 5.7. *The representation ρ_{kr} is irreducible if $\tau \neq 1$.*

When $\tau \neq 1$, the algebra $B_{G_m}(\Upsilon)$ have another irreducible representation ρ_{Kr} whose dimension is m. ρ_{Kr} isn't isomorphic to any ρ_i because $\rho_{Kr}(e_j) \neq 0$ but $\rho_i(e_j) = 0$. By Wedderburn-Artin theorem,

$$\dim B_{G_m}(\Upsilon) \geq \sum_{i=1}^{K} \dim \rho_i^2 + \dim \rho_{Kr}^2 = 2m + m^2.$$

We have proved the following theorem.

Theorem 5.2. *Suppose $\tau \neq 1$. The set T_m is a basis of $B_{G_m}(\Upsilon)$. In particular $\dim B_{G_m}(\Upsilon) = 2m + m^2$. $B_{G_m}(\Upsilon)$ is a semisimple algebra.*

It is known there is a cellular structure $(\Lambda, M, C, *)$ on $\mathbb{C}G_m$. The cellular structure $(\bar{\Lambda}, \bar{M}, \bar{C}, *)$ of $B_{G_m}(\Upsilon)$ is as follows.

- $\bar{\Lambda} = \Lambda \coprod \{\lambda_{Kr}\}$.We keep the original partial order in Λ, and for any $\lambda \in \Lambda$, let $\lambda_{Kr} \prec \lambda$.
- For $\lambda \in \Lambda$, $\bar{M}(\lambda) = M(\lambda)$. $\bar{M}(\lambda_{Kr}) = \{0, 1, \cdots, m-1\}$.
- For $\lambda \in \Lambda$, and $S, T \in \bar{M}(\lambda)$ $\bar{C}^{\lambda}_{S,T} = C^{\lambda}_{S,T}$. For $i, j \in \bar{M}(\lambda_{Kr}) = \{0, 1, \cdots, m-1\}$, $\bar{C}^{\lambda_{Kr}}_{i,j} = we_j$, such that $w(j) = i$. It is well-defined because w and ws_j are the only elements in G_m satisfying $w'(j) = i$. But $ws_j e_j = we_j$.
- Define $*$ be the involution as defined in Lemma 5.5

It isn't hard to prove

Theorem 5.3. *Above $(\bar{\Lambda}, \bar{M}, \bar{C}, *)$ is a cellular structure for $B_{G_m}(\Upsilon)$.*

$B_{G_m}(\Upsilon)$ with m being even. The following facts are true. For $0 \leq j < i \leq m-1$,

- if $i - j$ is even, let $k(i,j) = (i+j)/2$, $k(i,j)' = [(i+j+m)/2]$, then $\{k | s_k s_j s_k = s_i\} = \{k(i,j), k(i,j)'\}$.
- if $i - j$ is odd, then $\{k | s_k s_j s_k = s_i\} = \varnothing$.

The set R of reflections in G_m consists of two conjugacy classes. $R = R_0 \coprod R_1$, where $R_0 = \{s_0, s_2, \cdots, s_{m-2}\}$ and $R_1 = \{s_1, s_3, \cdots, s_{m-1}\}$. In the datum Υ, for convenience we denote μ_{s_i} as μ_i, and τ_{s_i} as τ_i. We have: $\mu_i = \mu_0$, $\tau_i = \tau_0$ if i is even; $\mu_i = \mu_1$, $\tau_i = \tau_1$ if i is odd. So the datum Υ consists of 4 parameters $\mu_0, \mu_1, \tau_0, \tau_1$. An explicit form of Definition 5.1 in these cases is as follows.

Definition 5.3. *The algebra $B_{G_m}(\Upsilon)$ is generated by s_0, s_1; $e_0, e_1, \cdots, e_{m-1}$ with the following relations.*

(1) $s_0^2 = s_1^2 = 1$.

(2) $(s_0 s_1)^m = 1$.

(3) $s_i e_i = e_i s_i$ *for all* i. *Where* $s_i = [s_1 s_0 \cdots]_{2i-1}$.

(4) $we_i w^{-1} = e_j$ *if* $ws_i w^{-1} = s_j$, *for* $w \in G$.

(5) $e_i^2 = \tau_0 e_i$ if i is even, and $e_i^2 = \tau_1 e_i$ if i is odd.

(6) $e_i e_j = 0$ if $i - j$ is odd.

(7) $e_i e_j = (\mu_{k(i,j)} s_{k(i,j)} + \mu_{k(i,j)'} s_{k(i,j)'}) e_j$ if $i - j$ is even.

Still by lemma 5.2, the algebra $B_{G_m}(\Upsilon)$ is spanned by the subset $T_m = G_m \coprod \{we_i | w \in G_m,\ 0 \leq i \leq m-1\}$. Relation (3) above implies that for every i, $|\{we_i | w \in G_m\}| \leq |G_m/\{1, s_i\}| = m$. So we have

$$\dim B_{G_m}(\Upsilon) \leq |T_m| \leq 2m + m \cdot m = 4l + 4l^2.$$

We use Wedderburn-Artin theorem again to show that for generic Υ, $\dim B_{G_m}(\Upsilon) \geq 4l + 4l^2$.

Let $\pi : B_{G_m}(\Upsilon) \to \mathbb{C}G_m$ be the natural projection sending all e_i's to 0. Through π, every irreducible representation of G_m induce an irreducible representation of $B_{G_m}(\Upsilon)$. Besides of these induced representations we have four more different representations described as follows. Two of them are the infinitesimal version of the generalized Krammer representation. Let's write down the flat connection for the generalized Krammer representation in present case. The relevant space is $V_{G_m} = \mathbb{C} < v_0, \cdots, v_{m-1} >$. The action ρ of G_m on V_{G_m} is

$$\rho(w)(v_i) = v_j, \text{ if } w s_i w^{-1} = s_j \text{ for } w \in G_m\ .$$

For every $0 \leq i \leq m-1$, we define a projector $p_i \in \mathrm{End}(V_{G_m})$ as follows. For $n \in \mathbb{Z}$ define $\epsilon(n) = 0$ if n is even, $\epsilon(n) = 1$ if n is odd.

$p_i(v_i) = \tau_0 v_i$ if i is even; $p_i(v_i) = \tau_1 v_i$ if i is odd.

$p_i(v_j) = (\mu_{\epsilon([\frac{i+j}{2}])} + \mu_{\epsilon([\frac{i+j}{2}+l])}) v_i$ if $i - j$ is even.

$p_i(v_j) = 0$ if $i - j$ is odd.

It isn't hard to prove the following lemma.

Lemma 5.8. *(1)The map $w \mapsto \rho(w)$ for $w \in G_m$; $e_i \mapsto p_i$ for $0 \leq i \leq m-1$ extends to a representation $\mathrm{Kr} : B_{G_m}(\Upsilon) \to \mathrm{End}(V_{G_m})$.*
(2) Let $V^0_{G_m} = \mathbb{C}\langle v_{2i} \rangle_{0 \leq i \leq l-1}$, and $V^1_{G_m} = \mathbb{C}\langle v_{2i+1} \rangle_{0 \leq i \leq l-1}$. Then $V_{G_m} = V^0_{G_m} \oplus V^1_{G_m}$ is a decomposition as $B_{G_m}(\Upsilon)$ representations. We denote the subrepresentation on $V^0_{G_m}$ as Kr^0, and the subrepresentation on $V^1_{G_m}$ as Kr^1.

Define two matrix $M^0 = (m^0_{i,j})_{l \times l}$ and $M^1 = (m^1_{i,j})_{l \times l}$ whose entries are determined by $p_{2(i-1)}(v_{2(j-1)})) = m^0_{i,j}(v_{2(i-1)})$ and $p_{2i-1}(v_{2j-1}) = m^1_{i,j} v_{2i-1}$.

Lemma 5.9. *If $\mathrm{Det}M^0 \neq 0$ ($\mathrm{Det}M^1 \neq 0$), then Kr^0 (Kr^1) are different irreducible representations of $B_{G_m}(\Upsilon)$. Both of them are different from those representations induced from π.*

Proof. For Kr^0, first every vector v_{2i} is a generator of the representation since G_m acts on the $\{v_{2i}\}_{0 \leq i \leq l-1}$ transitively. Then for any nonzero vector $v = \Sigma_{0 \leq i \leq l-1} \alpha_i v_{2i}$ in $V^0_{G_m}$, because $\mathrm{Det}M^0 \neq 0$ so there is some $0 \leq i \leq l-1$ such that $p_{2i}(v) = \lambda v_{2i} \neq 0$. So all nonzero vectors are generators and Kr^0 is irreducible. The case of Kr^1 can be proved similarly. Because in Kr^0 those elements p_{2i-1} 's act as zero but in Kr^1 they don't, so Kr_0 isn't isomorphic to Kr^1. In those representations induced from π, all the elements p_i's act as zero. So both Kr^0 and Kr^1 are different from them.

$\square$

There are two more representations K^0 and K^1 as follows. They arise from the left ideal generated by $(s_l e_0 - e_0)$ and the left ideal generated by $(s_{l+1} e_1 - e_1)$ respectively.

We equip an invariant Euclidean metric with $V \cong \mathbb{R}^2$ on which G_m acts. Then for every $0 \le i \le 2l-1$, we chose a unit vector ν_i perpendicular to H_i, let $\Phi = \{\pm\nu_i\}_{0\le i\le 2l-1}$. The set Φ is something like a root system.

Let W_{G_m} be a complex linear space with a basis $\{w_i\}_{0\le i\le 2l-1}$. For $0 \le i \le 2l-1$, define $\bar{s}_i \in \mathrm{End}(W_{G_m})$ as follows.

$\bar{s}_i(w_j) = \epsilon^i_{j,k} w_k$, if $s_i(\nu_{[j+l]}) = \epsilon^i_{j,k}\nu_{[k+l]}$.

Lemma 5.10. *The correspondence* $s_i \mapsto \bar{s}_i$ *for* $0 \le i \le 2l-1$ *extends to a representation of* G_m.

Proof. The identity $s_i(\nu_{[j+l]}) = \epsilon^i_{j,k}\nu_{[k+l]}$ implies $\epsilon^i_{j,k} = \epsilon^i_{k,j}$, so $\bar{s}_i^{\,2} = \mathrm{id}$. And since for every pair of $0 \le i,j \le m-1$, and every $0 \le k \le m-1$, $(s_i s_j)^{m_{i,j}}(\nu_k) = \nu_k$, so $(\bar{s}_i\bar{s}_j)^{m_{i,j}}(w_k) = w_k$. It follows $(\bar{s}_i\bar{s}_j)^{m_{i,j}} = \mathrm{id}$. So the representation is well defined.

For every $0 \le l \le 2l-1$, we define $p_i \in \mathrm{End}(W_{G_m})$ as follows.

$p_i(w_j) = 0$ if $2 \nmid i-j$.

$p_{2i}(\nu_{2j}) = (\mu_{i+j}\epsilon^{i+j}_{2j,2i} + \mu_{[i+j+l]}\epsilon^{[i+j+l]}_{2j,2i})\nu_{2i}$.

$p_{2i-1}(\nu_{2j-1}) = (\mu_{i+j-1}\epsilon^{i+j-1}_{2j-1,2i-1} + \mu_{[i+j+l-1]}\epsilon^{[i+j+l-1]}_{2j-1,2i-1})\nu_{2i-1}$.

Define two matrix $A^0 = (a^0_{i,j})_{l\times l}$, $A^1 = (a^1_{i,j})_{l\times l}$ whose entries are determined by $p_{2(i-1)}(w_{2(j-1)}) = a^0_{i,j} w_{2(i-1)}$, $p_{2i-1}(w_{2j-1}) = a^1_{i,j} w_{2i-1}$ for $1 \le i,j \le l$. We have

Theorem 5.4. *(1) The correspondence* $s_i \mapsto \bar{s}_i$, $e_i \mapsto p_i$ *for* $0 \le i \le 2l-1$ *extends to a representation of* $B_{G_m}(\Upsilon)$ *on* W_{G_m}.
(2) Let $W^0_{G_m} = \mathbb{C}\langle w_{2i}\rangle_{0\le i\le l-1}$, *and* $W^1_{G_m} = \mathbb{C}\langle w_{2i+1}\rangle_{0\le i\le l-1}$. *Then* $W_{G_m} = W^0_{G_m} \oplus W^1_{G_m}$ *is a decomposition of* $B_{G_m}(\Upsilon)$ *representations. We denote the representation on* $W^0_{G_m}$ $(W^1_{G_m})$ *as* K^0 (K^1).
(3) If $\det(A^0) \ne 0$ *and* $\det(A^1) \ne 0$, *then* K^0 *and* K^1 *are different irreducible representations of* $B_{G_m}(\Upsilon)$. *They are different from* Kr^0 *and* Kr^1,*and be different from those representations induced by* π *also.*

Proof. It is straightforward to see (2). For (1) we certify relations in $B_{G_m}(\Upsilon)$.

Now by Wedderburn-Artin theorem, when the data in Υ satisfies the conditions in lemma5.7 and theorem 5.4 (3), we have

$$\mathrm{dim}B_{G_m}(\Upsilon) \ge \#\{G_m\} + (\mathrm{dim}Kr^0)^2 + (\mathrm{dim}Kr^1)^2 + (\mathrm{dim}K^0)^2 + (\mathrm{dim}K^1)^2 = 4l + 4l^2.$$

We have proved the following theorem.

Theorem 5.5. *When the data in* Υ *satisfies the conditions in Lemma 5.7 and Theorem 5.4 (3), then* $B_{G_{2l}}(\Upsilon)$ *is a semisimple algebra having dimension* $4l + 4l^2$. *The subset* $G_{2l}\coprod\{we_i | w \in G_m,\ 0 \le i \le 2l-1\}$ *is a basis.*

Cellular structure for $B_{G_m}(\Upsilon)$. Still let $(\Lambda, M, C, *)$ be the known cellular structure on $\mathbb{C}G_{2l}$. We prove that $B_{G_{2l}}(\Upsilon)$ has a cellular structure $(\bar{\Lambda}, \bar{M}, \bar{C}, *)$ as follows.

- $\bar{\Lambda} = \Lambda \cup \{\lambda_{Kr^0}, \lambda_{Kr^1}, \lambda_{K^0}, \lambda_{K^1}\}$. We keep the partial order in Λ, and let $Kr^i \prec \lambda$, Kr^1 for any $\lambda \in \Lambda$ and any i. Let $K^i \prec Kr^j$ for any i, j.

- For $\lambda \in \Lambda$, $\bar{M}(\lambda) = M(\lambda)$.
- $\bar{M}(\lambda_{K_{r^0}}) = \bar{M}(\lambda_{K^0}) = \{0, 2, \cdots, 2l-2\}$.$\bar{M}(\lambda_{K_{r^1}}) = \bar{M}(\lambda_{K^1}) = \{1, 3, \cdots, 2l-1\}$.
- Denote the involution defined in Lemma 5.5 extending $s_i \mapsto s_i$, $e_i \mapsto e_i$ for $0 \le i \le 2l-1$ as $*$.
- Denote the including map $\mathbb{C}G_{2l} \to B_{G_{2l}}(\Upsilon)$ as j. For $\lambda \in \Lambda$, $S, T \in \bar{M}(\lambda)$, let $\bar{C}^{\lambda}_{S,T} = j(C^{\lambda}_{S,T})$.
- Let $[\cdots s_0 s_1]_i$ be the word having length i, and in which s_0, s_1 appear alternatively. Similarly define words $[\cdots s_1 s_0]_i$. Let $\bar{C}^{\lambda_{K^0}}_{2i,0} = [\cdots s_0 s_1]_i (s_l e_0 - e_0)$ for $0 \le i \le l-1$. Let $\bar{C}^{\lambda_{K^0}}_{0,2i} = *(\bar{C}^{\lambda_{K^0}}_{2i,0})$ for $0 \le i \le l-1$. Let $\bar{C}^{\lambda_{K^0}}_{2j,2i} = [\cdots s_0 s_1]_j (\bar{C}^{\lambda_{K^0}}_{0,2i})$ for $0 \le i, j \le l-1$.
- Let $\bar{C}^{\lambda_{K^1}}_{2i+1,1} = [\cdots s_1 s_0]_i (s_{l+1} e_1 - e_1)$ for $0 \le i \le l-1$. Let $\bar{C}^{\lambda_{K^1}}_{1,2i+1} = *(\bar{C}^{\lambda_{K^1}}_{2i+1,1})$ for any i. Let $\bar{C}^{\lambda_{K^1}}_{2j+1,2i+1} = [\cdots s_1 s_0]_j (\bar{C}^{\lambda_{K^1}}_{1,2i+1})$ for any i, j.
- The group G_m acts on $\{e_{2i}\}_{0 \le i \le l-1}$ by conjugacion as $w \cdot (e_{2i}) = w e_{2i} w^{-1}$. This action is transitive. Since $|G_m| = 4l$, we know for every pair of i, j, there are two classes in $G_m/\{1, s_{2j}\}$ sending e_{2j} to e_{2i}. Chose w_1, w_2 from each of these two classes, Let $\bar{C}^{\lambda_{K_{r^0}}}_{2i,2j} = \frac{1}{2}(w_1 + w_2) e_{2j}$. This definition is independent of the choice of w_1, w_2 since $s_{2j} e_{2j} = e_{2j}$. Similarly consider the conjugating action of G_m on $\{e_1, \cdots, e_{2l-1}\}$. There are two classes in $G_m/\{1, s_{2j-1}\}$ sending e_{2j-1} to e_{2i-1}. Chose w_1, w_2 from each of these two classes and let $\bar{C}^{\lambda_{K_{r^1}}}_{2i-1,2j-1} = \frac{1}{2}(w_1 + w_2) e_{2j-1}$.

The following result can be proved by direct computations.

Proposition 5.2. *Above data $(\bar{\Lambda}, \bar{M}, \bar{C}, *)$ define a cellular structure on $B_{G_{2l}}(\Upsilon)$.*

Remark 5.3. *From results as above it isn't hard to see that the quotient algebra $\bar{B}_{G_{2l}}(\Upsilon) = B_{G_{2l}}(\Upsilon)/ < (s_l e_0 - e_0), (s_{l+1} e_1 - e_1) >$ also satisfy our requests: generically semisimplity, cellularity, existence of a flat connection, implying generalized LK representation, but it has simpler structure than $B_{G_{2l}}(\Upsilon)$. So it can be a candidate for the version "generalized Brauer algebra " too. Such quotient algebra as $\bar{B}_{G_{2l}}(\Upsilon)$ only exist when the Dynkin diagram contains an even edge, or there is some entry of Coxeter matrix $m_{i,j} \in 2\mathbb{Z}$. For a general pseudo reflection group G we can define a quotient algebra $\bar{B}_G(\Upsilon)$ by adding a relation to Definition 5.1 as:*

(7). $se_i = s' e_i$ *if* $s(H_i) = s'(H_i) \neq H_i$ *for* $s, s' \in R$ *and* $i \in P$.

5.3 Canonical Presentations for Real and G(m,1,n) Cases

We define algebras $B'_G(\Upsilon)$ when G is a real or a cyclotomic reflection group of type $G(m, 1, n)$. These algebras have canonical presentations. Then we prove $B'_G(\Upsilon)$ is isomorphic to $B_G(\Upsilon)$. First we do it in cases of dihedral groups.

Definition 5.4. *The algebra $B'_G(\Upsilon)$ have the following presentation when G is the dihedral group of type $I_2(n)$ where $n = 2k+1$ is an odd number.*

- *Generators:* S_0, S_1, E_0, E_1.
- *Relations:* (1) $[S_0 S_1 \cdots]_n = [S_1 S_0 \cdots]_n$; (2) $S_0^2 = S_1^2 = 1$;

(3) $S_0 E_0 = E_0 = E_0 S_0$, $S_1 E_1 = E_1 = E_1 S_1$;

(4) $E_i^2 = \tau E_i$ *for* $i = 0, 1$;

(5) $E_0 [S_1 S_0 \cdots]_{2i-1} E_0 = \mu E_0$ *for* $1 \le i \le k$;

(6) $E_1 [S_0 S_1 \cdots]_{2i-1} E_1 = \mu E_1$ *for* $1 \le i \le k$;

(7) $[S_0 S_1 \cdots]_{2k} E_0 = E_1 [S_0 S_1 \cdots]_{2k}$;

When G is a dihedral group of type $I_2(2k)$, the element $s_0 s_k = [s_0 s_1 \cdots]_{2k}$ generate the center. We denote it as c. Denote the set of reflections in G as R.

Definition 5.5. *The algebra $B'_G(\Upsilon)$ have the following presentation when G is the dihedral group of type $I_2(n)$ when $n = 2k$ is even. We simply denote μ_{s_i}, τ_{s_i} as μ_i, τ_i respectively.*

- *Generators:* S_0, S_1, E_0, E_1.
- *Relations:* (1) $[S_0 S_1 \cdots]_n = [S_1 S_0 \cdots]_n$; (2) $S_0^2 = S_1^2 = 1$.

(3) $S_0 E_0 = E_0 S_0 = E_0$, $S_1 E_1 = E_1 S_1 = E_1$.

(4) $E_1 w E_0 = E_0 w E_1 = 0$ *for any word* w *composed by* S_0, S_1.

(5) $E_0 [S_1 S_0 \cdots]_{2i-1} E_0 = (\mu_i + \mu_{[i+k]} c) E_0$ *for* $1 \le i \le k$.

(6) $E_1 [S_1 S_0 \cdots]_{2i-1} E_1 = (\mu_i + \mu_{[i+k]} c) E_1$ *for* $1 \le i \le k$.

(7) $[S_1 S_0 \cdots]_{2k-1} E_0 = E_0 [S_1 S_0 \cdots]_{2k-1}$;

(8) $[S_0 S_1 \cdots]_{2k-1} E_1 = E_1 [S_0 S_1 \cdots]_{2k-1}$.

(9) $E_0^2 = \tau_0 E_0$, $E_1^2 = \tau_1 E_1$.

Theorem 5.6. *If G is a dihedral group, then $B_G(\Upsilon)$ is isomorphic to $B'_G(\Upsilon)$ defined above.*

Proof. We consider the cases when G is of type $I_2(2k)$. The cases for G of type $I_2(2k+1)$ are similar and easier.

Here we denote the algebra $B_G(\Upsilon)$ as B, and denote the algebra defined in theorem as B'. Let j be the morphism from $\mathbb{C}G$ to B' by mapping $s_i \in G$ to $S_i \in B'$ for $1 \le i \le n$. Let π be the morphism from B' to $\mathbb{C}G$ by mapping $S_i \in B'$ to $s_i \in G$, E_i to 0. It is easy to check these two morphisms are well defined. There is $\pi \circ j = id_{\mathbb{C}G}$, which implies j is injective. For saving notations we denote $j(w)$ as w for $w \in G$. By Theorem 5.1, there is also a natural injection l from $\mathbb{C}G$ to B. We will also denote $l(w)$ as w for $w \in G$, and we will identify S_i with s_i when we need.

For $2 \le 2i \le 2k-2$, choose any $w \in G$ such that $s_{2i} = w s_0 w^{-1}$ and let $E_{2i} = w E_0 w^{-1}$. E_{2i} is well defined with no dependence on choice of w. For example, choose $w = [S_1 S_0 \cdots]_{2i-1}$ so $E_{2i} = [S_1 S_0 \cdots]_{2i-1} E_0 [S_1 S_0 \cdots]_{2i-1}$. Similarly for $3 \le 2i-1 \le 2k-1$, define $E_{2i-1} = [S_1 S_0 \cdots]_{2i-2} E_1 [S_1 S_0 \cdots]_{2i-2} = [S_1 S_0 \cdots]_{2i-1} E_1 [S_1 S_0 \cdots]_{2i-1}$. Define a map ϕ from the set of generators of B to B' as : $\phi(s_0) = S_0$; $\phi(s_1) = S_1$; $\phi(e_i) = E_i$ for $0 \le i \le 2k-1$. Then ϕ extends to a morphism from B to B'. To prove it we only need to certify that ϕ keep all the relations in Definition 5.1. The cases of relation (1), (2) is straightforward. So ϕ can extend to a map from $\mathbb{C}G \cup \{e_0, \cdots, e_{2k-1}\}$. We prove the case of relation (7), others are similar. First consider the relation for $e_{2i} e_0$. We have

$\phi(e_{2i})\phi(e_0) = E_{2i} E_0 = [s_1 s_0 \cdots]_{2i-1} E_0 [s_1 s_0 \cdots]_{2i-1} E_0 = s_i E_0 s_i E_0 = s_i(\mu_i + \mu_{[i+k]} c) E_0$
$= (\mu_i s_i + \mu_{[i+k]} s_{[i+k]}) E_0 = \phi((\mu_{k(2i,0)} s_{k(2i,0)} + \mu_{k(2i,0)'} s_{k(2i,0)'}) e_0)$.

Since relation (7) is invariant under conjugation of $w \in G$, And for any $x \in \{s_0, s_1, e_0, \cdots, e_{2k-1}\}$ any $w \in G$, we have $\phi(w v w^{-1}) = w \phi(v) w^{-1}$, so the other relations for $e_{2i} e_{2j}$ reduce to the cases of $e_{2i} e_0$.

The relations for $e_{2i+1} e_{2j+1}$ are similar.

Then we define a morphism $\psi : B' \to B$ by extending the correspondence $S_0 \mapsto s_0$, $S_1 \mapsto s_1$, $E_0 \mapsto e_0$, $E_1 \mapsto e_1$. It is easy to check ψ is well-defined and be the inverse of ϕ.

$\square$

Suppose G_M is a finite Coxeter group with Coxeter matrix $M = (m_{i,j})_{n\times n}$. As in [17], the condition for M to be a coxeter matrix is: $m_{i,i} = 1$; $m_{i,j} = m_{j,i} \geq 2$ for $i \neq j$. The group G_M has the following presentation:

- Generators: $s_1, s_2, \cdots, s_n$;
- Relations: $[s_i s_j \cdots]_{m_{i,j}} = [s_j s_i \cdots]_{m_{i,j}}$ for $i \neq j$; $s_i^2 = 1$ for any i.

G_M can be realized as a group generated by reflections in some n dimensional linear space naturally, through the geometric representation. In the following we identify G_M with it's image through the geometric representation, and we denote G_M as G.

For $w \in G$, any expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ with minimal length is called a reduced form of w, and define the length of w as $l(w) = r$.

For the purpose of proving a key lemma we need to introduce the Artin group A_M with coxeter matrix M. It has the following presentation.

- Generators: $\sigma_1, \sigma_2, \cdots, \sigma_n$;
- Relations : $[\sigma_i \sigma_j \cdots]_{m_{i,j}} = [\sigma_j \sigma_i \cdots]_{m_{i,j}}$ for $i \neq j$.

Here we denote A_M as A. Let A^+ be the monoid generated with the same set of generators and relations. Let $J : A^+ \to A$ be the natural morphism of of monoids. We have

Theorem 5.7. *J is an injective map.*

The following theorem is well known.

Theorem 5.8. *For any $w \in G$, suppose $l(w) = r$ and let $s_{i_1} \cdots s_{i_r}$ and $s_{j_1} \cdots s_{j_r}$ be two reduced forms of w, then in A^+ we have $\sigma_{i_1} \cdots \sigma_{i_r} = \sigma_{j_1} \cdots \sigma_{j_r}$.*

So there is a well defined injective map $\tau : G \to A^+$ as follows. For $w \in G$, let $s_{i_1} \cdots s_{i_k}$ be a reduced form of w and let $\tau(w) = \sigma_{i_1} \cdots \sigma_{i_k}$. Denote the natural map from A^+ to G extending $\sigma_i \mapsto s_i$ as π.

In A^+ we denote $b \prec c$ if there is $a \in A^+$ such that $ab = c$. This define a partial order for A^+. Here is an important result in Artin group theory.

Theorem 5.9. [7], [14] *For $a \in A^+$, if $\sigma_i \prec a$, $\sigma_j \prec a$, then $[\cdots \sigma_j \sigma_i]_{m_{i,j}} \prec a$.*

Now we can prove the following lemma.

Lemma 5.11. *Suppose G acts on a set S. Suppose there is a subset $\{v_1, \cdots, v_n\}$ of S such that*

(1) If $m_{i,j} = 2k+1$, then $[s_i s_j \cdots]_{2k}(v_i) = v_j$;

(2) If $m_{i,j} = 2k$, then $[s_i s_j \cdots]_{2k-1}(v_j) = v_j$;

(3) If $m_{i,j} = 2$, then $s_i(v_j) = v_j$.

(4) $s_i(v_i) = v_i$.

Then an identity $w s_i w^{-1} = s_j$ in G implies $w(v_i) = v_j$.

Proof. We prove it by induction on $l(w)$. When $l(w) = 0$ it is evident. Suppose the lemma is true when $l(w) < k$ and suppose we have an identity $w s_i w^{-1} = s_j$ where $l(w) = k$. If $l(w s_i) = l(w) - 1$, let $w' = w s_i$. Since $w' s_i (w')^{-1} = w s_i w^{-1} = s_j$, by induction we have $w'(v_i) = v_j$. Which implies $w(v_i) = w'(v_i) = v_j$ by (4).

Now suppose $l(w s_i) = l(w) + 1$. Let $s_{i_1} \cdots s_{i_k}$ be a reduced form of w. We have $s_{i_1} \cdots s_{i_k} s_i = s_j s_{i_1} \cdots s_{i_k}$. Because both sides of the identity are reduced forms, by theorem5.8 we have $\sigma_{i_1} \cdots \sigma_{i_k} \sigma_i = \sigma_j \sigma_{i_1} \cdots \sigma_{i_k} = \tau(w s_i)$.

From the condition $l(ws_i) = l(w) + 1$ we know $i_k \neq i$, so by theorem5.9 we have $[\cdots \sigma_{i_k}\sigma_i]_{m_{i_k,i}} \prec \tau(ws_i)$. So $\tau(ws_i) = a[\cdots \sigma_{i_k}\sigma_i]_{m_{i_k,i}}$ for some $a \in A^+$. Denote $\pi(a)$ as w' ,and $\pi([\cdots \sigma_i\sigma_{i_k}]_{m_{i_k,i}-1})$ as u. So $w = w'u$. An argument of length shows $l(w') = l(w) - l(u)$. There is $us_iu^{-1} = s_{i_k}$, and by (1), (2) , (3) we have $u(v_i) = v_{i_k}$. So $w's_{i_k}(w')^{-1} = ws_iw^{-1} = s_j$. By induction we have $w'(v_{i_k}) = v_j$ which implies $w(v_i) = w'u(v_i) = v_j$.

□

Let R be the set of reflections contained in G. Let $\Upsilon = \{\mu_s, \tau_s\}_{s\in R}$ be a set of numbers satisfying $\mu_s = \mu_{s'}$, $\tau_s = \tau_{s'}$ if s and s' lie in the same conjugacy class. We have the following theorem.

Theorem 5.10. *The algebra $B_G(\Upsilon)$ has the following presentation.*

- *Generators:* $s_1, \cdots, s_n, e_1, \cdots, e_n$.
- *Relations:* *(1)*$s_i^2 = 1$ *for any* i;

(2) $[s_is_j\cdots]_{m_{i,j}} = [s_js_i\cdots]_{m_{i,j}}$ *for all* i, j;

(3) $s_ie_i = e_i = e_is_i$ *for any* i;

(4) $e_i^2 = \tau_ie_i$, *where* $\tau_i = \tau_{s_i}$;

(5) $s_ie_j = e_js_i$ *and* $e_ie_j = e_je_i$ *if* $m_{i,j} = 2$;

(6) $e_i[s_js_i\cdots]_{2l-1}e_i = \mu_{s_\epsilon}e_i$ *for* $1 \leq l \leq k$, *where* $m_{i,j} = 2k + 1$.

Where $\epsilon = i(j)$ *if* l *is odd (even).*

(7) $[s_is_j\cdots]_{2k}e_i = e_j[s_is_j\cdots]_{2k}$ *If* $m_{i,j} = 2k + 1$.

(8) $e_iwe_j = 0$ *for any word* w *composed from* $\{s_i, s_j\}$ *If* $m_{i,j} = 2k$.

(9) $e_i[s_js_i\cdots]_{2l-1}e_i = (\mu_s + \mu_{s'}c_{i,j})e_i$ *for* $1 \leq l \leq k$ *,If* $m_{i,j} = 2k$. *Where* $c_{i,j} = [s_is_j\cdots]_{2k}$, $s = [s_js_i\cdots]_{2l-1}$ *and* $s' = c_{i,j}^{-1}s$.

(10) $[s_is_j\cdots]_{2k-1}e_j = e_j[s_is_j\cdots]_{2k-1}$ *If* $m_{i,j} = 2k$.

Denote the algebra presented in theorem as $B'_G(\Upsilon)$. The strategy is to construct a morphism from $B'_G(\Upsilon)$ to $B_G(\Upsilon)$ and a morphism back, then prove they are the inverse of each other.

In definition 5.1, we first suppose G is the Coxeter group with Coxeter matrix M, then realize it as a reflection group through the geometric representation $\rho : G \to GL(V)$. We identify G with its image in $GL(V)$, denote $\rho(s_i)$ as s_i. Since G is real, the index set of reflection hyperplanes P are in one to one correspondence with the set of reflections R. So it is convenient to denote the reflection hyperplane of $s \in R$ as H_s and write e_i in the definition as e_s when $H_i = H_s$. Let $\mathcal{A} = \{H_s\}_{s\in R}$ be the set of reflection hyperplanes. Define $\phi(s_i) = s_i \in GL(V)$, $\phi(e_i) = e_{s_i}$ for $1 \leq i \leq n$.

Lemma 5.12. ϕ *extends to a morphism from* $B'_G(\Upsilon)$ *to* $B_G(\Upsilon)$.

Proof. We need to certify that ϕ satisfies all relations of $B'_G(\Upsilon)$. (1) to (4) are straightforward . When $m_{i,j} = 2$, The parabolic subgroup $G_{\{i,j\}}$ of G generated by s_i, s_j is $\mathbb{Z}_2 \times \mathbb{Z}_2$. By [15], the set $\{H_s \mid H_s \supseteq H_{s_i} \cap H_{s_j}\}$ are in one to one correspondence with reflections in $G_{\{i,j\}}$. Thus there are no other reflection hyperplanes containing $H_{s_i} \cap H_{s_j}$ except H_{s_i} and H_{s_j}. So by definition 5.1 (4), $\phi(e_i)\phi(e_j) = e_{s_i}e_{s_j} = e_{s_j}e_{s_i} = \phi(e_j)\phi(e_i)$, so relation (5) is satisfied.

When $m_{i,j} = 2k+1$, to certify that ϕ satisfies (6) it is enough to show $e_{s_i}[s_is_j\cdots]_{2l-1}e_{s_i} = \mu_\epsilon e_{s_i}$. Denote the reflection $[s_is_j\cdots]_{2l-1}$ as s. The identity is equivalent to $se_{s_i}se_{s_i} = \mu_\epsilon se_{s_i}$.

By (3) of definition 5.1, it is equivalent to $e_{ss_is}e_{s_i} = \mu_\epsilon s e_{s_i}$. By knowledge of dihedral groups , the only reflection conjugating s_i to ss_is is s. So the last identity is a consequence of definition5.1 (5). Because $[s_is_j\cdots]_{2k}s_i[s_is_j\cdots]_{2k}^{-1} = s_j$, so by Definition5.1(3), we have $[s_is_j\cdots]_{2k}e_{s_i} = e_{s_j}[s_is_j\cdots]_{2k}$. So (7) is also satisfied by ϕ.

When $m_{i,j} = 2k > 2$, s_i and s_j don't lie in the same conjugacy class. let w be any element in the parabolic subgroup of G generated by $\{s_i, s_j\}$. Denote ws_iw^{-1} as s. The reflection s lie in different conjugacy classes with s_j. Now $|\{H_t\ |H_t \supset H_s \cap H_{s_j}\ \}| = 2k > 2$, so conditions for definition5.1(9) are fulfilled and we have $e_se_{s_j} = 0$, which is equivalent to $e_{s_i}we_{s_i} = 0$. So ϕ satisfies (8).

The facts that ϕ satisfies (9),(10) can be proved similarly like the case of (6), (7).

Now we construct a morphism from $B_G(\Upsilon)$ to $B'_G(\Upsilon)$. First we need the following lemma. Let G acts on R by conjugation.

Through making quotient of $B'_G(\Upsilon)$ over the ideal generated by e'_is, it isn't hard to prove that the morphism J from $\mathbb{C}G$ to $B'_G(\Upsilon)$ by sending s_i to s_i is injective. So for $w \in G$ we identify $J(w)$ with w . And denote the imbedding image of R in $B'_G(\Upsilon)$ as R'. Set $E' = \{we_iw^{-1}\}_{w\in G, 1\le i\le n}$, $E = \{e_s\}_{s\in R}$. By definition of $B'_G(\Upsilon)$, the conjugating action of G on E' satisfies conditions in lemma5.10. So the map $e_i \mapsto e_{s_i}$ $(1 \le i \le n)$ extends to a G-equivariant bijection $\varphi : E' \to E$. Define a map $\psi : E \cap R \to E' \cap R'$ by $\psi(e_s) = \varphi^{-1}(e_s)$, $\psi(s) = s$. We have the following lemma.

Lemma 5.13. *The map ψ extends to a morphism from $B_G(\Upsilon)$ to $B'_G(\Upsilon)$. Still denote it as ψ.*

Proof. We prove that ψ satisfies relation (5) in definition5.1. The fact that ψ satisfies other relations can be proved similarly. Let $s_\alpha, s_\beta \in R$ such that $\{s \in R\ |H_s \supseteq H_{s_\alpha} \cap H_{s_\beta}\} \supsetneq \{s_\alpha, s_\beta\}$. Suppose the edge $L = H_{s_\alpha} \cap H_{s_\beta}$ is in the closure of some Weyl chamber Δ. Let Δ_0 be the fundamental Weyl chamber whose walls consist of $\{H_{s_1}, \cdots, H_{s_n}\}$. Since G acts on the set of Weyl chambers transitively, there exists $w \in G$ such that $w\Delta = \Delta_0$. So, there exists $w \in G$ and $1 \le i < j \le n$ such that $wL = H_{s_i} \cap H_{s_j}$.

Denote the parabolic subgroup of G generated by $\{s_i, s_j\}$ as $G_{i,j}$. Since $ws_\alpha w^{-1}$ and $ws_\beta w^{-1}$ fix every point in $H_{s_i} \cap H_{s_j}$, there exist $\dot{s}, \ddot{s} \in R \cap G_{i,j}$ such that $w\psi(e_{s_\alpha})w^{-1} = \dot{s}$, $w\psi(e_{s_\beta})w^{-1} = \ddot{s}$.

Denote the dihedral group of type $I_2(m_{i,j})$ as G'. Denote the set of reflections in G' as R'. Let ϕ be the injective morphism from G' to G by sending s_0, s_1 to s_i,s_j respectively. We consider the generalized algebra $B_{G'}(\Upsilon')$, where the data $\Upsilon' = \{\mu_s, \tau_s\}_{s\in R'}$ is determined by setting $\mu_s = \mu_{\phi(s)}$, $\tau_s = \tau_{\phi(s)}$ for $s \in R'$.

By the canonical presentation of $B_{G'}(\Upsilon')$ in theorem5.8 and theorem5.9, the morphism ϕ can be extended to a morphism from $B_{G'}(\Upsilon')$ to $B_G(\Upsilon)$ by mapping e_0, e_1 to e_i, e_j respectively. We still denote this morphism as ϕ. Let $\dot{r} = \phi^{-1}(\dot{s})$, $\ddot{r} = \phi^{-1}(\ddot{s})$. In $B_{G'}(\Upsilon')$ we have $e_{\dot{r}}e_{\ddot{r}} = (\Sigma_{r\in R':r\dot{r}r=\ddot{r}}\mu_r r)e_{\ddot{r}}$. Apply ϕ to this identity we get

$$e_{\dot{s}}e_{\ddot{s}} = (\Sigma_{r\in R':r\dot{r}r=\ddot{r}}\mu_r\phi(r))e_{\ddot{s}} = (\Sigma_{s\in R:s\dot{s}s=\ddot{s}}\mu_s s)e_{\ddot{s}}.$$

The second equality is because $\mu_{\phi(r)} = \mu_r$, and

$$\{\phi(r)\}_{r\in R':r\dot{r}r=\ddot{r}} = \{s \in R \cap G_{i,j} \mid s\dot{s}s = \ddot{s}\} = \{s \in R \mid s\dot{s}s = \ddot{s}\}.$$

So by conjugating above identity by w^{-1} we have

$$\psi(e_{s_\alpha})\psi(e_{s_\beta}) = w^{-1}e_{\dot{s}}e_{\ddot{s}}w = (\Sigma_{s\in R:s\dot{s}s=\ddot{s}}\mu_s w^{-1}sw)e_{s_\beta} = (\Sigma_{s\in R:ss_\alpha s=s_\beta}\mu_s s)e_{s_\beta}.$$

By definition ψ and ϕ are apparently the inverse of each other, so theorem5.8 is proved. When M is a finite type Coxeter matrix of simply laced type, any two reflections in G_M are conjugate to each other. So in the data Υ, $\mu_s = \mu$, $\tau_s = \tau$ for all $s \in R$. Using lemma5.3 we can let $\mu = 1$. So $B_{G_M}(\Upsilon)$ can be also denoted as $B_{G_M}(\tau)$. Those relations in theorem5.8 in these cases are actually the same as the relations of generalized Brauer algebras in [10].

Remark 5.4. *The algebra $B'_G(\Upsilon)$ can be defined for Coxeter groups of infinite type as well.*

The Cyclotomic $G(m,1,n)$ cases. Let G be the cyclotomic pseudo reflection group of type $G(m,1,n)$. As in [5], let V be a n-dimensional complex linear space with a positive definite Hermitian metric $<,>$, let $\{v_1,\cdots,v_n\}$ be a orthonormal base. Then G can be imbedded in $U(V)$. It's image consists of monomial matrices whose entries are m-th roots of unit. Here we give a concise description of some facts of G without proof. Suppose $(z_1,\cdots,z_n)$ is the coordinate system corresponding to $\{v_1,\cdots,v_n\}$. Let $\xi = \exp(\frac{2\pi\sqrt{-1}}{m})$. For $i \neq j$, $0 \le a \le m-1$, define $H_{i,j;a} = \ker(z_i - \xi^a z_j) = (v_i - \xi^a v_j)^\perp$. Define $H_i = \ker(z_i) = (v_i)^\perp$. Then $H^a_{i,j} = H_{j,i;-a}$.

Let $s_{i,j;a} \in U(V)$ be the unique reflection fixing every points in $H^a_{i,j}$. Let s_i be the pseudo reflection defined by: $s_i(v_i) = \xi v_i$; $s_i(v_j) = v_j$ for $j \neq i$. Then the set $\mathcal{A}$ of reflection hyperplanes of G is $\{H_{i,j;a}\}_{i<j;0\le a\le m-1} \cup \{H_i\}_{1\le i\le n}$. The set of pseudo reflections R of G is

$$\{s_{i,j;a}\}_{i<j;0\le a\le m-1} \amalg (\amalg_{k=1}^{m-1}\{s_i^k\}_{1\le i\le n}).$$

The left side of above identity gives a decomposition of R into conjugacy classes. It is well known that G has a canonical presentation also as follows.

Proposition 5.3. *If set $S_0 = s_1$; $S_i = s_{i,i+1;0}$ for $1 \le i \le n-1$, we have the following canonical presentation of G.*

- *Generators:* $S_0,\cdots,S_{n-1}$.
- *Relations:* $S_0^m = S_i^2 = id$ *for* $1 \le i \le n-1$; $S_iS_{i+1}S_i = S_{i+1}S_iS_{i+1}$ *for* $1 \le i \le n-2$; $S_iS_j = S_jS_i$ *for* $|i-j| \ge 2$; $S_0S_1S_0S_1 = S_1S_0S_1S_0$.

Now we have a look at the algebra $B_G(\Upsilon)$. The data Υ now essentially consists of $\mu,\mu_1,\cdots,\mu_{m-1},\tau_0,\tau_1$. Where $\mu_{s_{i,j;a}} = \mu$, $\mu_{s_i^k} = \mu_k$; $\tau_{H_{i,j;a}} = \tau_1$, $\tau_{H_i} = \tau_0$.

The following is a explicit definition of $B_G(\Upsilon)$ according to definition5.1.

Definition 5.6. *Here is the explicit definition of $B_G(\Upsilon)$ when G is the pseudo reflection group of type $G(m,1,n)$.*

- *Generators:* $\{\bar{w}\}_{w\in G} \cup \{e_{i,j;a}\}_{i<j;0\le a\le m-1} \cup \{e_i\}_{1\le i\le n}$.
- *Relations: (0)* $\bar{w}_1\bar{w}_2 = \bar{w}_3$ *if* $w_1w_2 = w_3$ *in* G;

(1) $\bar{s}_{i,j;a}e_{i,j;a} = e_{i,j;a}\bar{s}_{i,j;a} = e_{i,j;a}$; $s_ie_i = e_is_i = e_i$;

(2) $(e_{i,j;a})^2 = \tau_1e_{i,j;a}$; $(e_i)^2 = \tau_0e_i$;

(3) $\bar{w}e_{i,j;a} = e_{k,l;b}\bar{w}$ *if* $w(H_{i,j;a}) = H_{k,l;b}$; $\bar{w}e_i = e_j\bar{w}$ *if* $w(H_i) = H_j$;

(4) $e_{i,j;a}e_{k,l;b} = e_{k,l;b}e_{i,j;a}$ *if* $\{i,j\}\cap\{k,l\} = \emptyset$; $e_{i,j;a}e_k = e_ke_{i,j;a}$ *if* $k \notin \{i,j\}$;

(5) $e_{i,j;a}e_{i,k;b} = \mu\bar{s}_{j,k;b-a}e_{i,k;b}$ *for* $j \neq k$;
$e_i e_j = (\Sigma_{1\leq a\leq m-1}\mu_a \bar{s}_{i,j;a})e_j$;
(6) $e_{i,j;a}e_i = e_i e_{i,j;a} = 0$.

As in the real case, we define the following algebra $B'_G(\Upsilon)$ and prove it is isomorphic to $B_G(\Upsilon)$.

Definition 5.7. *The algebra* $B'_G(\Upsilon)$ *is defined with the following generators and relations.*

- *Generators:* $S_0, S_1, \cdots, S_{n-1}$, $E_0, E_1 \cdots, E_{n-1}$;
- *Relations:* (1) $S_0^m = S_i^2 = \mathrm{id}$ *for* $1 \leq i \leq n-1$; $S_0S_1S_0S_1 = S_1S_0S_1S_0$;

$S_iS_j = S_jS_i$ *for* $|i-j| \geq 2$; $S_iS_{i+1}S_i = S_{i+1}S_iS_{i+1}$ *for* $1 \leq i \leq n-1$.
(2) $E_0^2 = \tau_0 E_0$; $E_i^2 = \tau_1 E_i$ *for* $1 \leq i \leq n-1$;
(3) $S_1(S_0)^iS_1E_0 = E_0S_1(S_0)^iS_1$ *for any* i; $S_iE_0 = E_0S_i$ *for* $i \geq 1$;
$(S_0)^iS_1(S_0)^i(E_1) = E_1(S_0)^iS_1(S_0)^i$ *for any* i.
$S_iS_{i+1}E_i = E_{i+1}S_iS_{i+1}$ *for* $i \geq 1$; $S_iE_j = E_jS_i$ *for* $|i-j| \geq 2$.
(4) $S_iE_i = E_i = E_iS_i$.
(5) $E_1S_0^iE_1 = \mu_iE_1$ *for* $1 \leq i \leq m-1$; $E_0S_1E_0 = (\Sigma_{i=1}^{m-1}\mu S_1S_0^iS_1S_0^{-i})E_0$.
(6) $E_iE_j = E_jE_i$ *for* $|i-j| \geq 2$; $E_iE_{i+1} = \mu S_iS_{i+1}S_iE_{i+1}$.
(7) $E_0WE_1 = E_1WE_0 = 0$ *for any word* W *of* S_i*'s.*

Theorem 5.11. *The algebra* $B'_G(\Upsilon)$ *is isomorphic to* $B_G(\Upsilon)$.

Proof. The strategy of proof of this theorem is the same as for the real case. We construct a morphism Φ from $B'_G(\Upsilon)$ to $B_G(\Upsilon)$ and a morphism Ψ in reverse direction. Once these morphisms are constructed, it is straight forward to see they are inverse of each other so the theorem is proved. The morphism Φ is constructed by setting

$$\Phi(S_0) = s_1; \Phi(S_i) = s_{i,i+1;0} \text{ for } i \geq 1; \Phi(E_0) = e_1; \Phi(E_i) = e_{i,i+1,0} \text{ for } i \geq 1.$$

It isn't hard to certify that Φ satisfying all relations in definition5.5, so Φ can extend to a morphism. To define Ψ, the main step is still the definition of $\Psi(e_i)$ and $\Psi(e_{i,j;a})$. The following lemma shows there are well defined elements F_i's and $F_{i,j;a}$'s such that if we set

$$\Psi(w) = w, \Psi(e_i) = F_i, \Psi(e_{i,j;a}) = F_{i,j;a},$$

Then Ψ can extend to a morphism from $B_G(\Upsilon)$ to $B'_G(\Upsilon)$ by certifying that it all relations in definition5.4. The proof is almost the same as in proof of lemma5.12 so we skep it.

∎

Construction of F_i and $F_{i,j;a}$ The following lemma is similar to lemma5.10.

Lemma 5.14. *Let* G *be the pseudo reflection group of type* $G(m,1,n)$. *Suppose* G *acts on a set* S *and suppose there is a subset* $\{v_0, v_1, \cdots, v_{n-1}\}$ *such that:*
(1) $(S_0)^iS_1(S_0)^i(v_1) = v_1$ *for any* i; $S_1(S_0)^iS_1(v_0) = v_0$ *for any* i.
(2) $S_iS_{i+1}(v_i) = v_{i+1}$ *for* $i \geq 1$; $S_iS_{i-1}(v_i) = v_{i-1}$ *for* $i \geq 2$.
(3) $S_i(v_j) = v_j$ *if* $|i-j| \geq 2$.
(4) $S_i(v_i) = v_i$.

Where S_i*'s are generators of* G *as in proposition5.3. For convenience of presentation here we denote* H_1 *as* $\mathbb{H}_0$, $H_{i,i+1;0}$ *as* $\mathbb{H}_i$ *for* $i \geq 2$. *Then the identity* $w(\mathbb{H}_i) = \mathbb{H}_j$ *implies* $w(v_i) = v_j$, *where* $w \in G$.

Proof. In this case instead of using Artin monoid we prove it by direct computation. Let $\bar{v}_i = S_{i-1}\cdots S_1(v_0)$ for $1 \le i \le n$; $\bar{v}^a_{i,j} = (S_{j-1}\cdots S_1 S_0 S_1 \cdots S_{j-1})^a S_{j-1}\cdots S_{i+1}(v_i)$ for $i < j-1$; $\bar{v}^a_{i,i+1} = (S_i \cdots S_0 \cdots S_i)^a(v_i)$. The following identities show that the set $\{\bar{v}_i\}_{1\le i\le n} \cup \{\bar{v}^a_{i,j}\}_{i<j}$ is closed under the action of G, and the map $J : \mathcal{A} \to \{\bar{v}_i\}_{1\le i\le n} \cup \{\bar{v}^a_{i,j}\}_{i<j}$:

$$H_{i,j;a} \mapsto \bar{v}^a_{i,j}; \quad H_i \mapsto \bar{v}_i$$

is G equivariant. Thus our theorem is proved.

(a) $S_0(\bar{v}_i) = \bar{v}_i$.

$S_0(\bar{v}_i) = S_{i-1}\cdots S_2 S_0 S_1(v_0) = S_{i-1}\cdots S_2 S_1 \cdot S_1 S_0 S_1(v_0) = S_{i-1}\cdots S_2 S_1(v_0) = \bar{v}_i$.

(b) $S_0(\bar{v}^a_{1,i}) = \bar{v}^{a-1}_{1,i}$.

$S_0(\bar{v}^a_{1,i}) = S_0(S_{i-1}\cdots S_0 \cdots S_{i-1})^a S_{i-1}\cdots S_2(v_1) = (S_{i-1}\cdots S_0\cdots S_{i-1})^a S_{i-1}\cdots S_2 S_0(v_1)$
$= (S_{i-1}\cdots S_0\cdots S_{i-1})^{a-1} S_{i-1}\cdots S_1 S_0 S_1 S_0(v_1) = (S_{i-1}\cdots S_0\cdots S_{i-1})^{a-1} S_{i-1}\cdots S_2(v_1)$
$= \bar{v}^{a-1}_{1,i}$.

(c) $S_0(\bar{v}^a_{i,j}) = \bar{v}^a_{i,j}$ if $i \ge 2$.

$S_0(\bar{v}^a_{i,j}) = S_0(S_{j-1}\cdots S_1 S_0 S_1\cdots S_{j-1})^a S_{j-1}\cdots S_{i+1}(v_i)$
$= (S_{j-1}\cdots S_1 S_0 S_1\cdots S_{j-1})^a S_{j-1}\cdots S_{i+1} S_0(v_i) = \bar{v}^a_{i,j}$.

(d) $S_i(\bar{v}_i) = \bar{v}_{i+1}$ for $i \ge 1$.

$S_i(\bar{v}_i) = S_i S_{i-1}\cdots S_1(v_0) = \bar{v}_{i+1}$.

(e) $S_i(\bar{v}_{i+1}) = \bar{v}_i$ for $i \ge 1$.

Equivalent to (d).

(f) $S_i(\bar{v}_j) = \bar{v}_j$ if $i \ne 0$ and $j \ne i, i+1$.

$j \ne i, i+1 \Leftrightarrow i > j$ or $i < j-1$. If $i > j$ then $S_i(\bar{v}_j) = S_i S_{j-1}\cdots S_1(v_0)$
$= S_{j-1}\cdots S_1 S_i(v_0) = \bar{v}_j$; If $i < j-1$ then $S_i(\bar{v}_j) = S_i S_{j-1}\cdots S_1(v_0)$
$= S_{j-1}\cdots S_i S_{i+1} S_i \cdots S_1(v_0) = S_{j-1}\cdots S_{i+1} S_i S_{i+1}\cdots S_1(v_0)$
$= S_{j-1}\cdots S_1 S_{i+1}(v_0) = \bar{v}_j$.

(g) $S_i(\bar{v}^a_{i,l}) = \bar{v}^a_{i+1,l}$ if $l \ge i+2$.

First we have

$S_i(S_{l-1}\cdots S_0\cdots S_{l-1}) = S_{l-1}\cdots S_i S_{i+1} S_i \cdots S_0 \cdots S_{l-1}$
$= S_{l-1}\cdots S_{i+1} S_i S_{i+1}\cdots S_0\cdots S_{l-1} = S_{l-1}\cdots S_0 \cdots S_{i+1} S_i S_{i+1}\cdots S_{l-1}$
$= (S_{l-1}\cdots S_0\cdots S_{l-1})S_i$.

So

$S_i(\bar{v}^a_{i,l}) = S_i(S_{l-1}\cdots S_0\cdots S_{l-1})^a S_{l-1}\cdots S_{i+1}(v_i)$
$= (S_{l-1}\cdots S_0\cdots S_{l-1})^a S_i S_{l-1}\cdots S_{i+1}(v_i)$
$= (S_{l-1}\cdots S_0\cdots S_{l-1})^a S_{l-1}\cdots S_{i+2} S_i S_{i+1}(v_i)$
$= (S_{l-1}\cdots S_0\cdots S_{l-1})^a S_{l-1}\cdots S_{i+2}(v_{i+1}) = \bar{v}^a_{i+1,l}$.

(h) $S_i(\bar{v}^a_{l,i}) = \bar{v}^a_{l,i+1}$ for $l < i$.

$S_i(\bar{v}^a_{l,i}) = S_i(S_{i-1}\cdots S_0\cdots S_{i-1})^a S_{i-1}\cdots S_{l+1}(v_l)$
$= (S_i\cdots S_0\cdots S_i)^a S_i S_{i-1}\cdots S_{l+1}(v_l) = \bar{v}^a_{l,i+1}$.

(i) $S_i(\bar{v}^a_{k,l}) = \bar{v}^a_{k,l}$ if $\{k,l\}\cap\{i,i+1\} = \emptyset$.

$S_i(\bar{v}^a_{k,l}) = S_i(S_{l-1}\cdots S_0\cdots S_{l-1})^a S_{l-1}\cdots S_{k+1}(v_k)$

□

5.4 A variation of $B_G(\Upsilon)$

In definition5.1 of the algebra $B_G(\Upsilon)$, if we replace the relation (6) with the following weaker relation

$(6)'$. $e_ie_j = e_je_i$, if $\{k \in I | H_k \supset H_i \cap H_j\} \neq \{i,j\}$ and $R(i,j) = \emptyset$,

we obtain an algebra $\hat{B}_G(\Upsilon)$ larger than $B_G(\Upsilon)$. It isn't hard to certify that $\hat{B}_G(\Upsilon)$ is also finite dimensional, the connection Ω as in proposition5.1 is still flat and G-invariant, and $\hat{B}_G(\Upsilon)$ still contain the generalized Krammer representations. In the following we explain the reason that taking $B_G(\Upsilon)$ as the generalized Brauer algebra but not $\hat{B}_G(\Upsilon)$.

Two important features of the Brauer algebra $\mathcal{B}_n(\tau)$ are that for generic τ the algebra $\mathcal{B}_n(\tau)$ is semisimple and that for all τ the algebra $\mathcal{B}_n(\tau)$ have the same dimension. We hope that generalized Brauer algebras should also hold these features. The following two lemmas show that these features don't hold for $\hat{B}_G(\Upsilon)$.

Here we consider the cases when G are dihedral groups. Notations are from section 5.2. In dihedral group G of type $I_2(2k+1)$, for any two reflection $s_i,\ s_j$ the set $R(i,j) \neq \emptyset$, so in the corresponding algebra $B_G(\Upsilon)$ the condition (6) doesn't appear. we only consider the cases when G is a dihedral group of type $I_2(2k)$. The main result of this section is the following proposition.

Proposition 5.4. *(1) Let* G *be dihedral group of type* $I_2(2n)$*. Let* H_i, H_j *be a pair of reflection hyperplanes such that* $R(i,j) = \emptyset$*, and* $s_is_j \neq s_js_i$*. Suppose* $s_is_js_i = s_k$*. In the algebra* $\hat{B}_G(\Upsilon)$*, if* Υ *satisfies the condition:* $\mu_i \pm \mu_{[n+i]} - \tau_k \neq 0$*, then for any irreducible representation* (V,ρ) *of* $\hat{B}_G(\Upsilon)$ *we have* $\rho(e_i)\rho(e_j) = \rho(e_j)\rho(e_i) = 0$*.*

(2) Let G *be dihedral group of type* $I_2(2n)$*. Let* H_i, H_j *be a pair of reflection hyperplanes such that* $R(i,j) = \emptyset$*, and* $s_is_j = s_js_i$*. In the algebra* $\hat{B}_G(\Upsilon)$*, if* Υ *satisfies the condition* $\tau_0(\mu_1 \pm \mu_0) \neq 0$ *and* $\mu_2 \pm \mu_{[n+2]} - \tau_4 \neq 0$*, then for any irreducible representation* (V,ρ) *of* $\hat{B}_G(\Upsilon)$ *we have* $\rho(e_i)\rho(e_j) = \rho(e_j)\rho(e_i) = 0$*.*

Proof. (1) First we have $e_ie_j = s_ie_ie_j = s_ie_je_i = s_ie_js_ie_i = e_ke_i$. The second equality is by using relation $(6)'$.

So $e_k(e_ie_j) = (e_k)^2e_i = \tau_k(e_ke_i) = \tau_ke_ie_j$. On the other hand,

$e_k(e_ie_j) = (e_ke_j)e_i = (\mu_is_i + \mu_{[n+i]}s_{[n+i]})e_je_i$.

Now $c = s_is_{[n+i]}$ is central in $\hat{B}_G(\Upsilon)$, so in the irreducible representation (V,ρ), $\rho(c)$ is a constant. Since $c^2 = 1$, so this constant is 1 or -1. When $\rho(c) = 1$, $(\mu_i\rho(s_i) + \mu_{[n+i]}\rho(s_{[n+i]}))\rho(e_j)\rho(e_i) = (\mu_i\rho(s_i) + \mu_{[n+i]}\rho(s_i))\rho(e_i)\rho(e_j) = (\mu_i + \mu_{[n+i]})\rho(e_j)\rho(e_i)$. So $(\mu_i + \mu_{[n+i]} - \tau_k)\rho(e_i)\rho(e_j) = 0$ and the condition in statement (1) imply $\rho(e_i)\rho(e_j) = 0$. Similarly when $\rho(c) = -1$, we have $(\mu_i - \mu_{[n+i]} - \tau_k)\rho(e_i)\rho(e_j) = 0$ and the condition in (1) implies $\rho(e_i)\rho(e_j) = 0$.

(2) By checking action of reflections on the set of reflection hyperplanes it isn't hard to see the cases in statement (2) only happen when $n = 2k+1$ and $j = [i+2k+1]$. First we consider the case $i = 0,\ j = 2k+1$. Remember the central element $c = s_is_{[2k+1+i]} = s_0s_{2k+1}$ for any i, so

$$s_i(e_0e_{2k+1}) = s_{[2k+1+i]}s_0s_{2k+1}e_0e_{2k+1} = s_{[2k+1+i]}s_0s_{2k+1}e_{2k+1}e_0 = s_{[2k+1+i]}e_0e_{2k+1}.$$

Now let (V,ρ) be an irreducible representation of $\hat{B}_G(\Upsilon)$ in which c acts as id_V. On the one hand $\rho[(\mu_0s_{2k+2} + \mu_1s_1)e_2s_1(e_0e_{2k+1})] = \rho[(\mu_0s_{2k+2} + \mu_1s_1)s_1(e_0)^2e_{2k+1}] = \rho[\tau_0(\mu_0 +$

$\mu_1)e_0e_{2k+1}]$. Where the second identity is by using relations of $B_G(\Upsilon)$ and by using above identity.

On the other hand $\rho[(\mu_0 s_{2k+2}+\mu_1 s_1)e_2 s_1(e_0e_{2k+1})] = \rho[e_0e_2s_1(e_0e_{2k+1})] = \rho[s_1e_2e_0e_0e_{2k+1}] = \rho[\tau_0 s_1 e_2 e_0 e_{2k+1}] = \rho[\tau_0 s_1 e_2 e_{2k+1} e_0]$. Since $R(2, 2k+1) = \emptyset$, and $s_2 s_{2k+1} \neq s_{2k+1} s_2$, so by statement (1) and by conditions in (2), since $s_2 s_{2k+1} s_2 = s_4$, we have $\rho(e_2 e_{2k+1}) = 0$. So $\tau_0(\mu_0+\mu_1)\rho(e_0e_{2k+1}) = \tau_0\rho(s_1e_2e_{2k+1}e_0) = 0$. The condition in statement (2) then implies $\rho(e_0)\rho(e_{2k+1}) = 0$.

$\square$

Remark 5.5. *Canonical presentations of* $B_G(\Upsilon)$ *may help to find possible deformations of these algebras, i.e, the generalized BMW algebras. We also hope to find* $B_G(\Upsilon)$ *in some other settings, for example, find geometric constructions of these algebras.*

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